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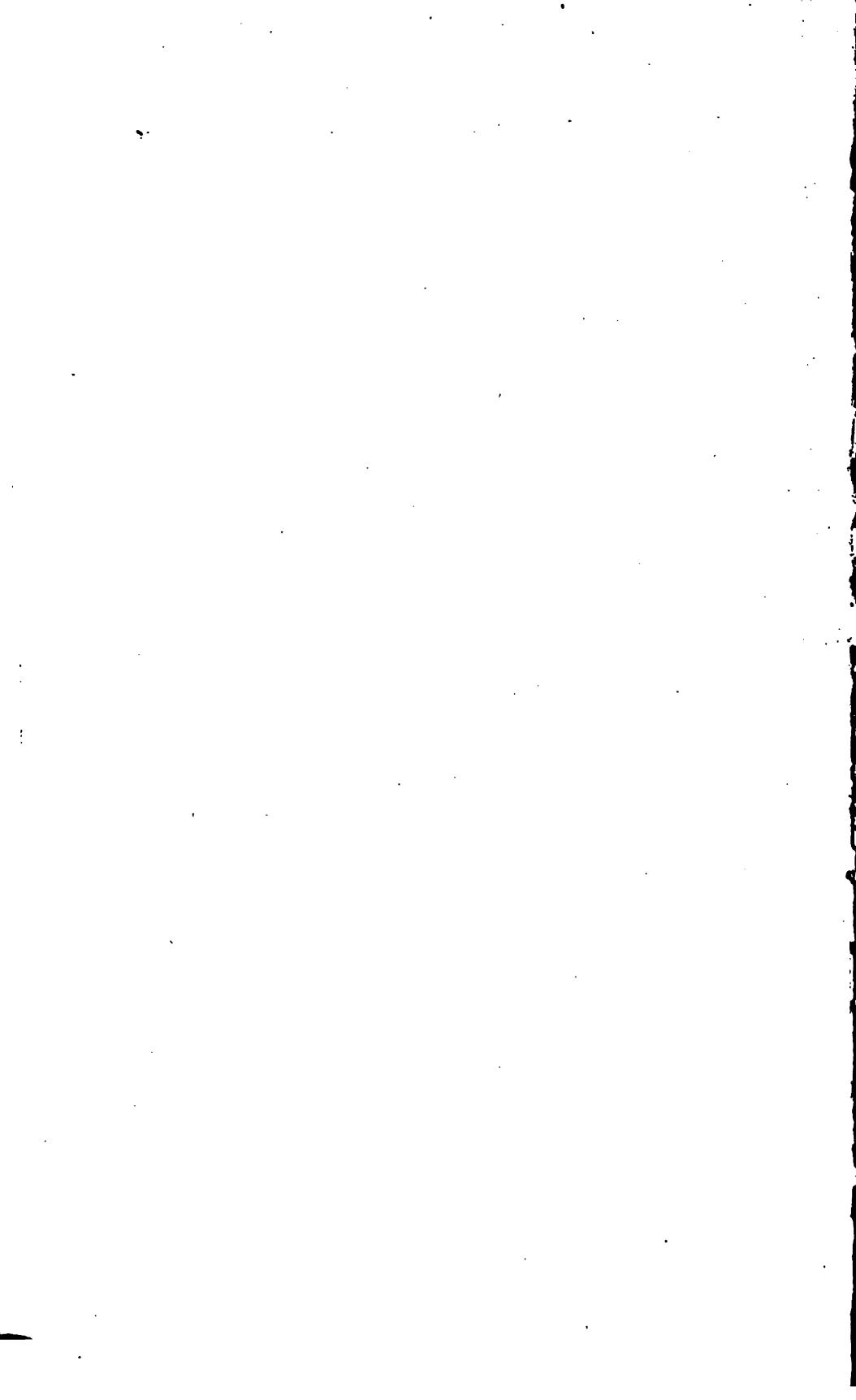
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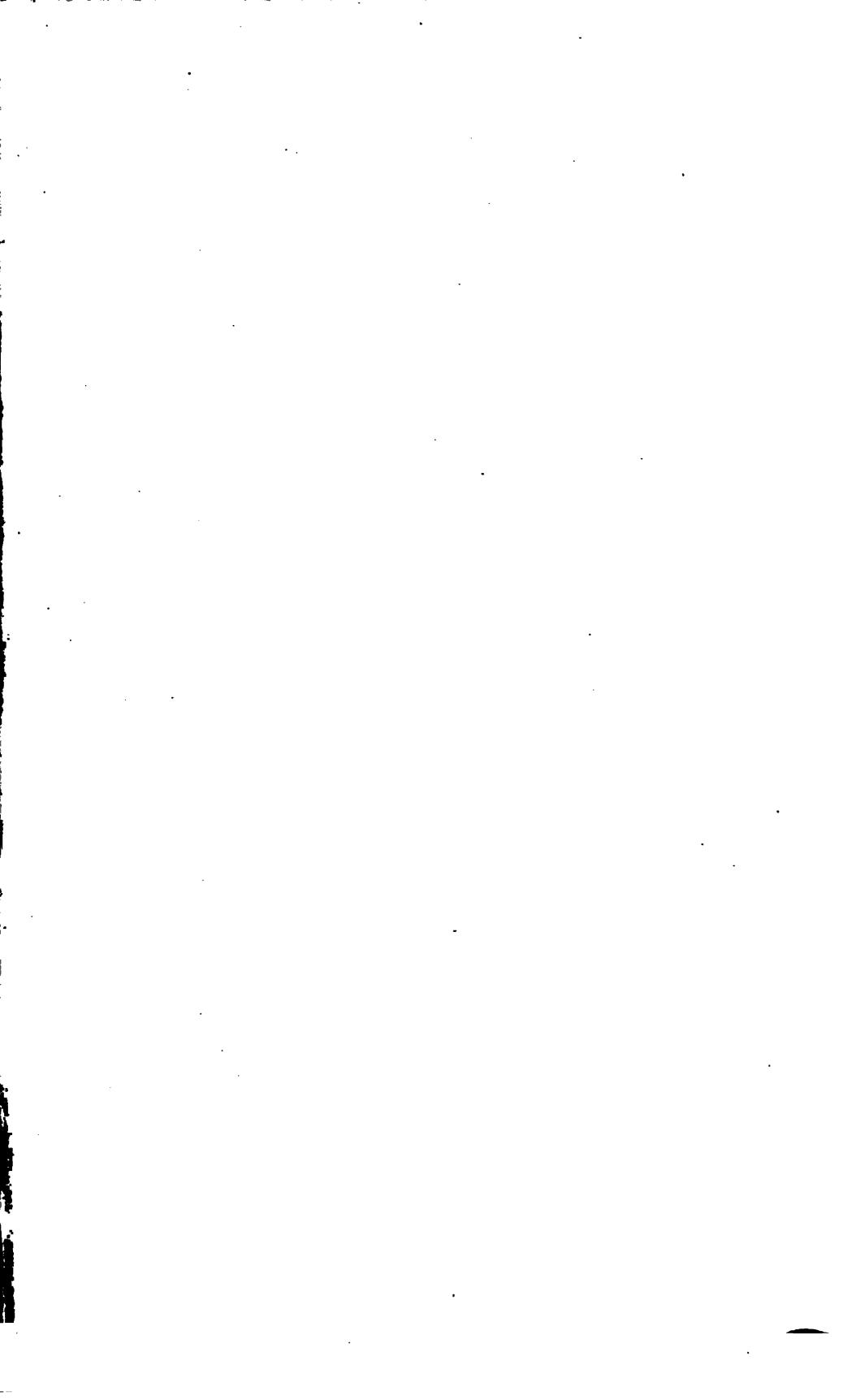
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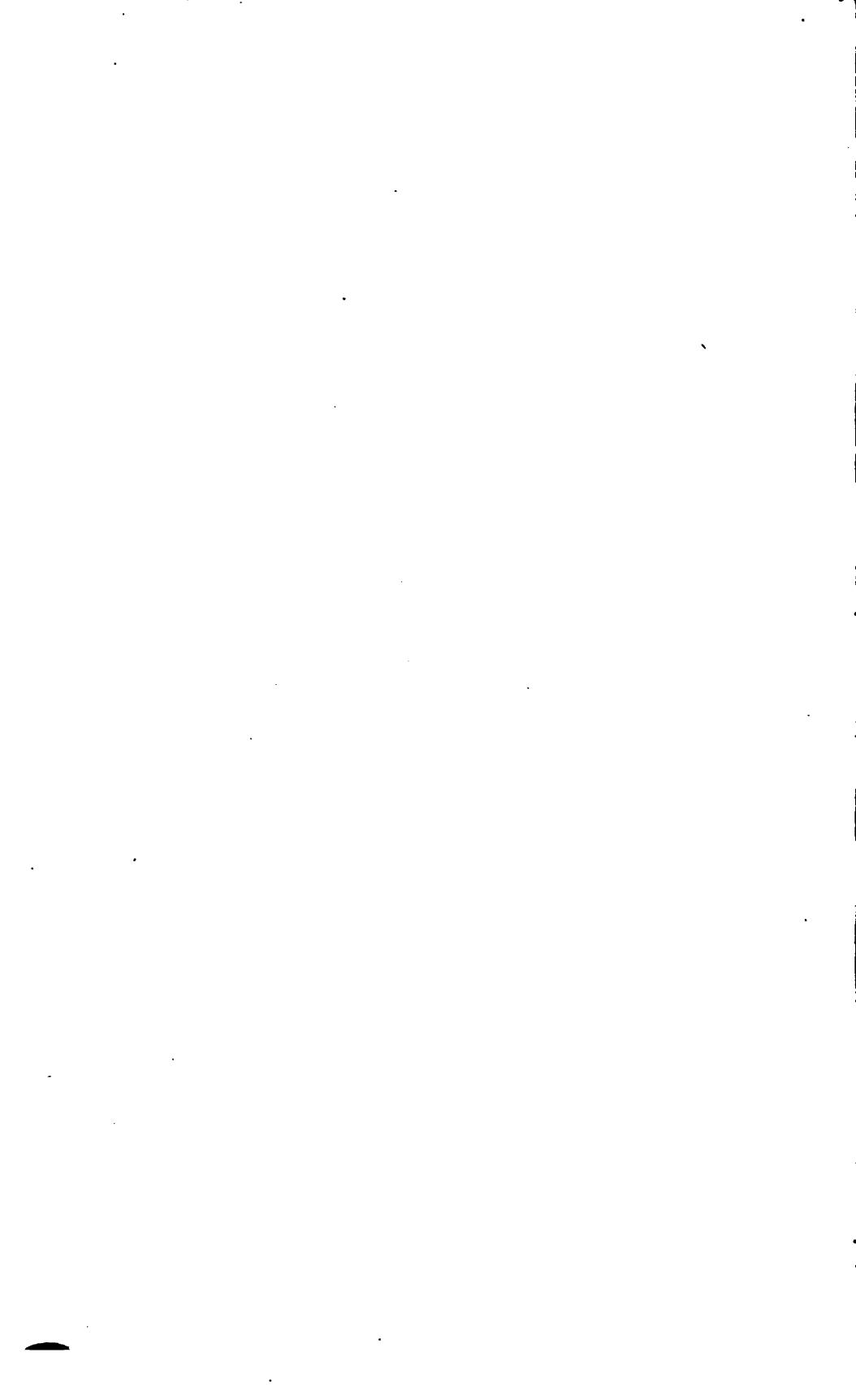
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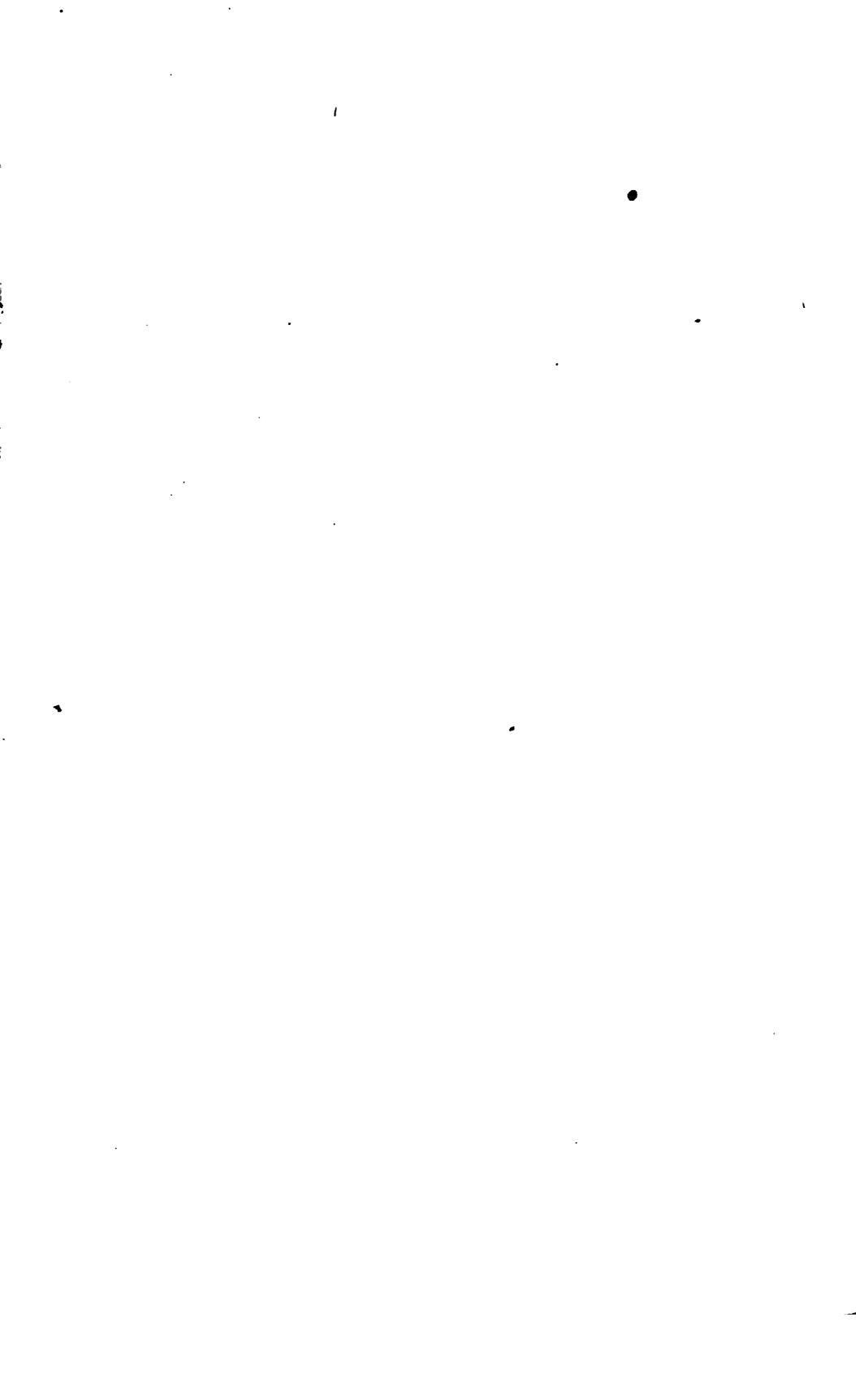
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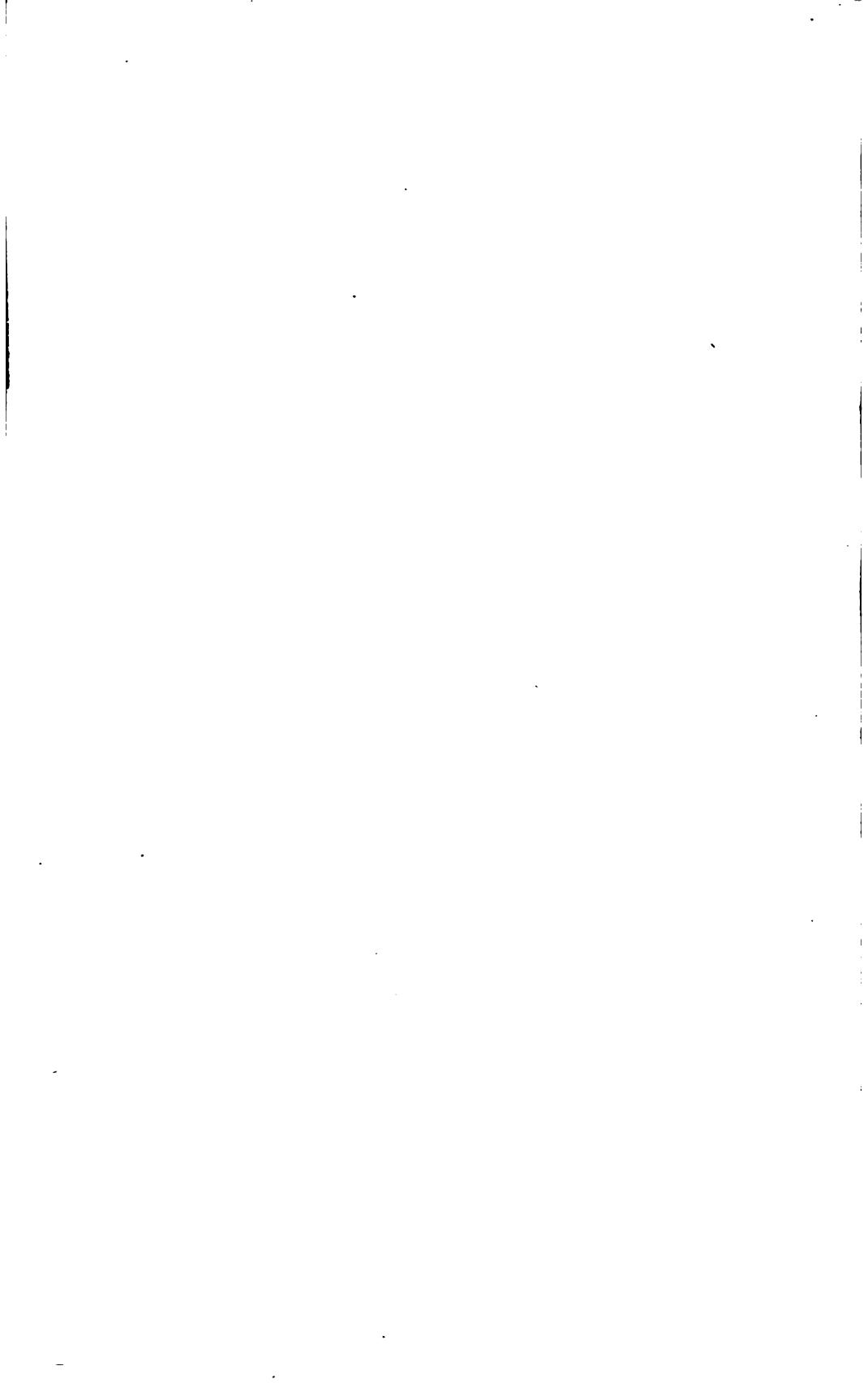
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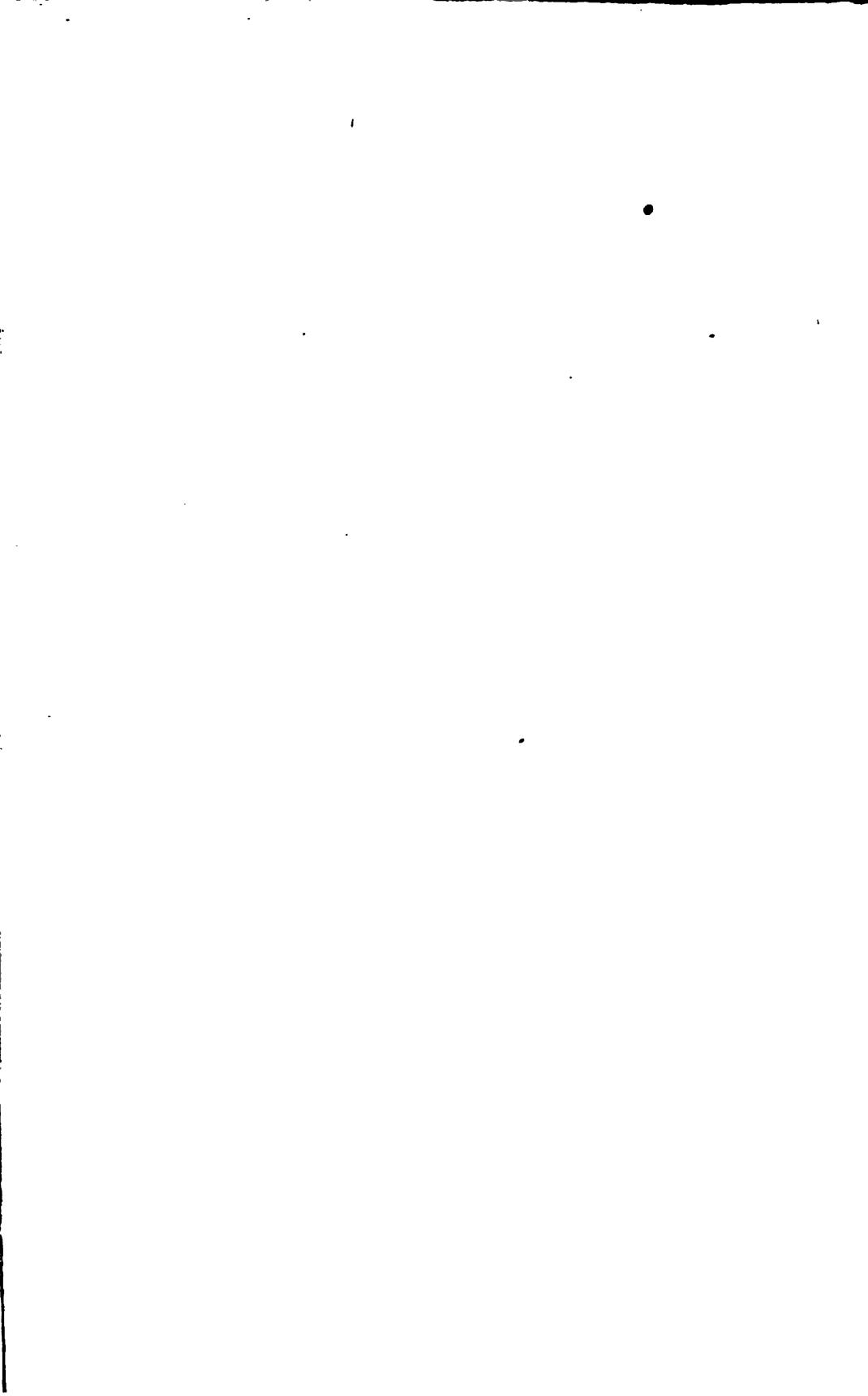














# MECHANICS OF THE GIRDER:

## A Treatise on Bridges and Roofs,

IN WHICH THE NECESSARY AND SUFFICIENT WEIGHT OF THE STRUCTURE IS CALCULATED,

NOT ASSUMED;

AND

THE NUMBER OF PANELS AND HEIGHT OF GIRDER THAT RENDER THE BRIDGE WEIGHT LEAST, FOR A GIVEN SPAN, LIVE LOAD, AND WIND PRESSURE, ARE DETERMINED.

BY

JOHN DAVENPORT CREHORE, C.E.

"Inveniam viam aut faciam."

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## PREFACE.

THE "Mechanics of the Girder" is presented to the public in an unfinished condition, just as it was left at the author's death, in October, 1884. All that then remained to be done was to carry out an example in each of the twelve classes of girders in a manner similar to that of the Brunel Girder in Class I. (Sections 2 and 3, Chapter X.), and the Double Parabolic Bow and Post Truss in Class II. (Chapter XI.). Of all these, the Post Truss promised to yield the most prolific results; and it may be possible, before another edition is published, to complete this calculation at least, if not to introduce other examples from the later classes. However, the a priori method of the author is fully set forth previous to the tenth chapter; and it is believed that no one else has as yet published any so satisfactory results from this method, if, indeed, the method has been hitherto attempted with any degree of success.

The author's family feel deeply grateful to Professor John N. Stockwell for his kindness in devoting much of his valuable time to the supervision of the proof-reading, for the many suggestions he has given during the publication, and particularly for his offer to conduct the work of completing the remaining examples. At his own suggestion, however, it has been thought expedient to delay no longer the publication of the completed portion of the book, and to leave any additional matter to be inserted later.

WILLIAM W. CREHORE.

July 29, 1886.

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## MECHANICS OF THE GIRDER.

## CHAPTER I.

#### PRESSURES IN ONE PLANE.

I. Force is a cause which changes, or tends to change, the condition of matter as to rest or motion. Whether there is or is not, in fact, any difference between force and pressure, it is sufficient for the purposes of this volume to treat them as identical, since it is with their measurable effects alone that we are here concerned.

A force is said to be given when its point of application, its direction, its line of action, and its intensity are known. Two pressures are equal which, acting on the same point, along the same line, and in opposite directions, neutralize each other; and, if two equal pressures act at the same point in the same direction, the result of their combined action is twice that of each separate pressure.

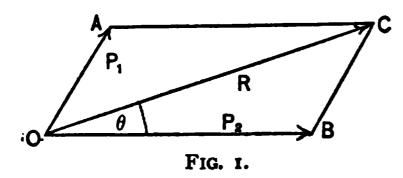
Pressures, therefore, may be compared by means of numbers expressing their intensities. Since the intensity of any one of the pressures to be compared may be taken as the standard, it follows that the unit pressure is entirely arbitrary, and may be a finite or an infinitesimal pressure.

When pressures are expressed by symbols, such as P, Q, R, etc., it is to be understood that these letters stand for num-

bers denoting the number of times the concrete unit is taken. Otherwise, such an expression as  $P^2$ , being the square of a concrete pressure, would be unintelligible.

A force or pressure may be conveniently represented by a geometrical straight line; one end of the line denoting the point of application of the force, the direction of the line being coincident with the direction of the force, and the number of linear units in the line being equal to the number of force units to be represented.

- 2. When many pressures act at the same time on a material particle, the result of their combined action is generally a definite pressure in a definite direction. This definite pressure is called the *resultant* of the acting or impressed pressures; and these latter, with reference to the resultant, are styled *components*. When the resultant is zero, the pressures are said to be in *equilibrium*; when the resultant of the given pressures is not zero, equilibrium may evidently be produced by introducing a new force which shall neutralize this resultant.
- 3. Parallelogram of Forces. It is shown in elementary works on mechanics, that if two forces act upon a single point,



and their intensities and directions be represented by two adjacent sides of a parallelogram, then the diagonal of the parallelogram drawn to the intersection of those two sides will

represent, both in magnitude and direction, the resultant of the two given forces.

If  $P_1$  and  $P_2$ , Fig. 1, are two forces acting at the point O, represented in magnitude and direction by the lines OA and OB, then, completing the parallelogram, AOBC, the resultant will be represented, in magnitude and direction, by the diagonal OC = R. When, therefore, a force is applied at O equal in intensity to R, and acting in the same line but in the opposite

direction, it will balance the given forces  $P_1$ ,  $P_2$ , and the three forces will be in equilibrium.

4. Triangle of Forces. — Since, in Fig. 1,  $AO = BC = P_1$ , the three sides of the triangle BOC (or AOC) represent, in magnitude and direction, three forces,  $P_1$ ,  $P_2$ , and R, which, acting in one plane on a given point, are in equilibrium; the direction of the forces being that of a point traversing the perimeter of the triangle. In this manner the value of the resultant may be constructed.

A formula for the value of R is found by solving the triangle of forces, where two sides and the angle included between them are given. Thus, if c = AOB = the angle between the given lines of action of  $P_1$  and  $P_2$ , we have, from the geometry of the figure, putting  $BOC = \theta$  (theta),

$$R^2 = P_1^2 + P_2^2 + 2P_1P_2\cos c. \tag{1}$$

$$\sin \theta = \frac{P_1}{R} \sin c. \tag{2}$$

Example. — Let  $P_1 = 8$ ,  $P_2 = 12$ ,  $c = 75^{\circ}$ . Then

$$R^2 = 8^2 + 12^2 + 2 \times 8 \times 12 \cos 75^\circ = 208 + 192 \times 0.25882$$
  
= 257.6933.  
 $R = 16.053$ .

 $\sin \theta = \frac{8}{16.053} \times 0.96593 = 0.48471.$ 

 $\theta = 28^{\circ} 59' 40''$ .

If the lines of action of the two forces,  $P_1$ ,  $P_2$ , are at right angles to each other,  $\cos c$  becomes zero, and equation (1) reduces to  $R^2 = P_1^2 + P_2^2$ , where R is the hypothenuse, and  $P_1$  and  $P_2$  are the other sides of a right-angled triangle.

Example. — When 
$$P_1 = 8$$
, and  $P_2 = 12$ ,

$$R^2 = 8^2 + 12^2 = 208$$
.  $R = 14.422$ 

In this case, Fig. 2, we have

$$P_{1} = R \quad \sin \theta = P_{2} \tan \theta.$$

$$P_{2} = R \quad \cos \theta = P_{1} \cot \theta.$$

$$R = P_{1} + \sin \theta = P_{1} \csc \theta.$$

$$R = P_{2} + \cos \theta = P_{2} \sec \theta.$$

$$A = P_{1} + \cos \theta = P_{2} \sec \theta.$$

$$A = P_{2} + \cos \theta = P_{3} \sec \theta.$$

$$A = P_{1} + \cos \theta = P_{2} \sec \theta.$$

$$A = P_{2} + \cos \theta = P_{3} \cot \theta.$$

5. Resolution of a Force. — Conversely, any force, R, acting at a given point with known intensity and direction, may be resolved into two component forces acting at the same point, having definite intensities and directions. Manifestly also may each one of the two components be resolved into two components, and so on without limit.

FIG. 2.

Example. — Resolve the force R = 100 tons, acting at the point O, Fig. 2, in the direction OC, into its horizontal and vertical components;  $\theta$  being equal to  $28^{\circ}$  59' 40".

From (3),

$$F_1 = R \sin \theta = 100 \times 0.48471 = 48.471 \text{ tons.}$$

$$P_2 = R \cos \theta = 100 \times 0.87467 = 87.467$$
tons.

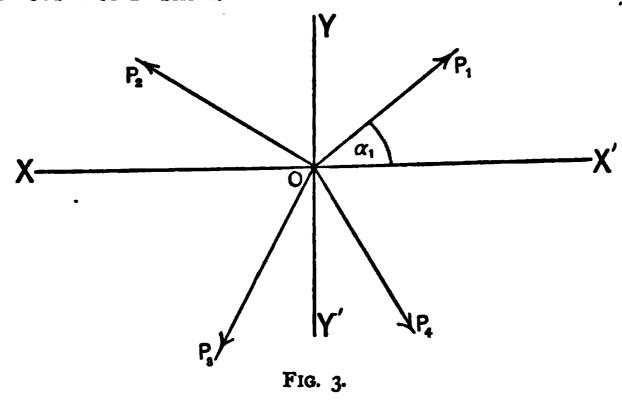
6. Resolution of Many Forces acting in One Plane at a Given Point. — Let there be any number of forces,  $P_1$ ,  $P_2$ ,  $P_3$ , etc., Fig. 3, acting in the plane of the axes XX', YY', at their point of intersection, O; and let  $\alpha$  (alpha) symbolize the angle between the line of action of any force and the axis of x.

Resolving each force into its horizontal and vertical components, and calling the sum of the horizontal components X, and the sum of the vertical components Y, these results:

$$X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \ldots = \sum P \cos \alpha, \quad (4)$$

$$Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \ldots = \Sigma P \sin \alpha; \quad (5)$$

the symbol  $\Sigma$  (sigma) denoting the sum of the terms having the form  $P \cos \alpha$  or  $P \sin \alpha$ .



7. Thus, for all the given forces acting in their various directions on the point O have been substituted two other forces, X and Y, acting at the same point; the one horizontally, the other vertically, and in the plane of the original forces. Now, if R is the resultant of the two forces X and Y, it must also be the resultant of the forces  $P_1$ ,  $P_2$ ,  $P_3$ , etc.; and,  $\theta$  being the angle between the resultant and the axis of x, we shall have

$$R\cos\theta = X = \Sigma P\cos\alpha, \tag{6}$$

$$R \sin \theta = Y = \sum P \sin \alpha, \qquad (7)$$

$$\therefore R^2 = X^2 + Y^2, \qquad (8)$$

$$\tan\theta = \frac{Y}{X}.$$
 (9)

When the given forces are in equilibrium, the resultant vanishes, and

$$X = \sum P \cos \alpha = 0. \tag{10}$$

$$Y = \sum P \sin \alpha = 0. \tag{11}$$

EXAMPLE. — Let 
$$P_1 = 10$$
 tons,  $\alpha_1 = 40^{\circ}$ .  
 $P_2 = 20$  tons,  $\alpha_2 = 150^{\circ}$ .  
 $P_3 = 30$  tons,  $\alpha_3 = 250^{\circ} = -110^{\circ}$ .  
 $P_4 = 40$  tons,  $\alpha_4 = 300^{\circ} = -60^{\circ}$ .

Required the intensity, R, and the direction,  $\theta$ , of the resultant.

$$X = 10 \cos 40^{\circ} + 20 \cos 150^{\circ} + 30 \cos 250^{\circ} + 40 \cos 300^{\circ} = 10 \cos 40^{\circ}$$

$$- 20 \cos 30^{\circ} - 30 \cos 70^{\circ} + 40 \cos 60^{\circ} = 10 \times 0.76604 - 20$$

$$\times 0.86603 - 30 \times 0.34202 + 40 \times 0.5 = 0.0792 \text{ tons.}$$

$$V = 10 \sin 40^{\circ} + 30 \sin 150^{\circ} + 30 \sin 250^{\circ} + 40 \sin 200^{\circ} = 10 \sin 40^{\circ}$$

$$Y = 10 \sin 40^{\circ} + 20 \sin 150^{\circ} + 30 \sin 250^{\circ} + 40 \sin 300^{\circ} = 10 \sin 40^{\circ} + 20 \sin 30^{\circ} - 30 \sin 70^{\circ} - 40 \sin 60^{\circ} = 10 \times 0.64279 + 20 \times 0.5 - 30 \times 0.93969 - 40 \times 0.86603 = -46.404 tons.$$

$$\tan \theta = \frac{-46.404}{0.0792} = -585.909, \quad \theta = -89^{\circ} 54' 8''.$$

$$R = [(0.0792)^2 + (-46.404)^2]^{\frac{1}{2}} = Y + \sin \theta = 46.404065 \text{ tons.}$$

The resultant is therefore in the fourth quadrant, and makes an angle of 5' 52'' with the axis of y.

This substitution of one force for many others is called the composition of forces.

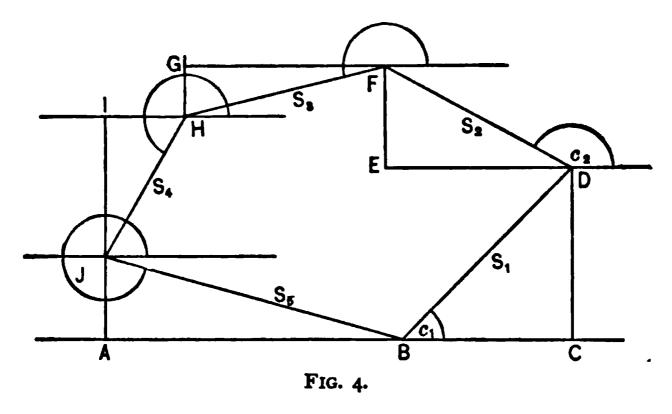
8. Polygon of Forces. — Let  $S_1$ ,  $S_2$ ,  $S_3$ , etc., in Fig. 4, be the five sides of a closed polygon. Measure the inclination of each side to the horizon, as indicated in the figure, for  $c_1$ ,  $c_2$ , etc.; then the sum of the horizontal projections of all the

sides is, in accordance with the trigonometrical signs of the cosine, found to be

$$\sum S \cos c = S_1 \cos c_1 + S_2 \cos c_2 + S_3 \cos c_3 + S_4 \cos c_4 + S_5 \cos c_5 = 0. \quad (12)$$
Since
$$S_1 \cos c_1 = +BC, \qquad S_2 \cos c_2 = -DE,$$

$$S_5 \cos c_5 = +AB, \qquad S_3 \cos c_3 = -FG,$$

$$S_4 \cos c_4 = -HI.$$



Equation (12) is true whatever be the number of sides of the polygon; and from its analogy to equation (10), viz.,

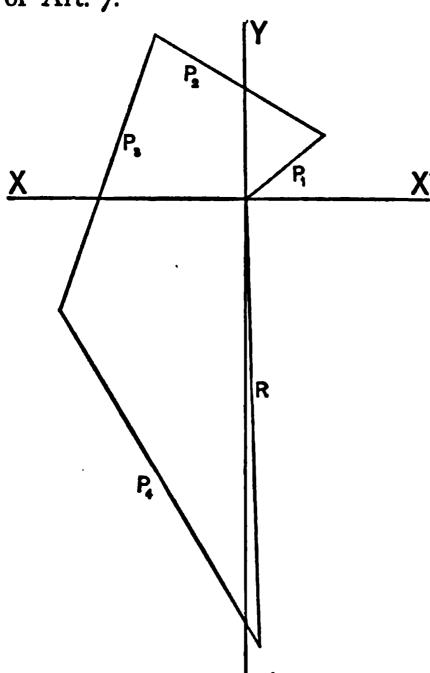
$$\Sigma P \cos \alpha = 0$$
,

we may enunciate the proposition, that when any number of forces acting at the same point, with their lines of action in the same plane, are in equilibrium, then the given forces may be represented, in magnitude and direction, by the sides of a closed polygon; the direction being, for each side, that of a point traversing the perimeter.

This proposition enables us to construct the resultant of many forces acting on a point in the common plane of their

lines of action, by regarding the unknown resultant, with its direction changed, as the side required to complete or close the polygonal figure due to the given forces.

Example. — Let us apply this proposition to the example of Art. 7.



This solution consists simply in drawing (Fig. 5) a continuous figure made up of lines proportional to the given forces and respectively parallel to their given lines of action, and each force having the direction a point would take in traversing the broken line from end to end. The straight line joining the two ends of this broken line will be the resultant sought, with its direction reversed.

It will be seen that the values of Y and R in this example are very nearly equal, and that the solution by construction can show an appreciable difference

in them, only when a large scale is used. In practice, however, either solution is accurate enough; and one serves to check the other.

The triangle of forces is a particular case of the closed polygon.

### CHAPTER II.

#### MOMENT OF A FORCE.

9. The moment of a force is the effect of the force's effort to turn the body on which it acts about a given point, and is measured by the number expressing the force, multiplied by the number denoting the perpendicular distance from the given point to the line of action of the force.

Moments, therefore, may be added and subtracted, and represented by lines, like other numbers.

Since the unit of the force and the unit of the perpendicular distance are arbitrary, it is usual to express the moment as a denominate number, designating both the units. Thus, 20 foot-tons, or ton-feet, means that the moment 20 is equivalent to the effect of a force or pressure of 20 tons acting at the perpendicular distance, or lever arm, of 1 foot from the axis of rotation, or to a force of 1 ton acting at the distance of 20 feet from the same axis.

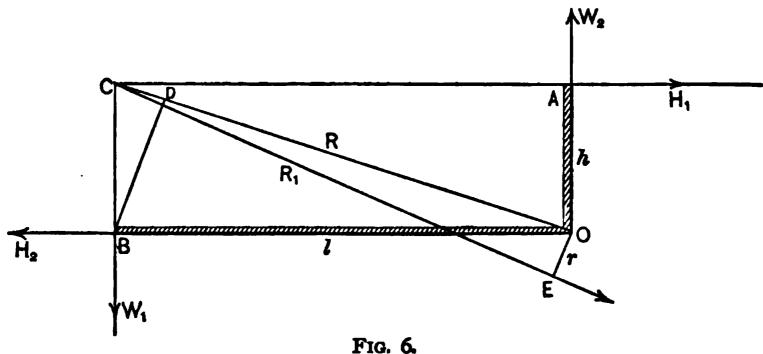
It is plain that the moment of a given force acting at a given perpendicular distance from the axis of rotation may be replaced by any one of an infinite number of equivalent moments.

10. In former articles forces have been considered as acting in straight lines in one plane and on a single point; tending in their united action, when the resultant does not vanish, to move that point or material particle in the direction of the resultant. Hence such forces are termed forces of translation.

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But, in the case now under consideration, we have two forces in one plane acting at two points in a rigid body, the one force at one point tending to turn the body about the other point.

I say two forces, for it is manifest, that, at the point which is taken as the centre of rotation, there must be a resistance to motion equal and opposite to the rotatory or tangential force acting at the other point. Such a system of two parallel forces acting in opposite directions is called a couple, and the perpendicular drawn to the lines of action of the forces is called the arm of the couple.



In Fig. 6, let AOB be a rigid body, beam, or bent lever, whose weight is not now to be taken into account; and let AO be perpendicular to OB, O being a fixed point about which the applied forces,  $W_{\rm r}$  and  $H_{\rm r}$ , acting at right angles to their respective lever arms, tend to turn the beam. Take OB = l, and OA = l, where l and l represent each a number of linear units. Then the moment of the force  $W_{\rm r}$  is  $-W_{\rm r}l$ , and the moment resulting from the action of  $H_{\rm r}$  is  $H_{\rm r}l$ ; giving them different signs because they tend to turn the beam in opposite directions about the point O.

If these two moments are equal, we have

$$H_i h - W_i l = 0, (13)$$

۲,

which shows that there is no rotation about the point O, and that the three forces  $W_1$ ,  $H_2$ , and the resistance to translation offered at O, are in equilibrium.

At the fixed point O are developed two forces: the one,  $W_2$ , equal to  $W_1$ , but acting in the opposite direction, OA; the other,  $H_2$ , equal to  $H_1$ , acting at O in the direction OB Now, these two forces, acting at the same point, O, must, by Art. 3, have a resultant equal to the diagonal of the rectangle of which  $W_2$  and  $H_2$  are the adjacent sides. Therefore the resultant  $R = \sqrt{W_2^2 + H_2^2}$ , and the tangent of the angle between the line of R and the line OB is  $\tan \theta = \frac{W_2}{H_2} = \frac{h}{l}$ , in the case of equilibrium, or when the two forces are inversely proportional to their lever arms, as shown in equation (13).

This resultant, R, is a force of translation, and may be graphically found by producing the lines of action of W, and H, till they intersect at C; then, if AC represent the intensity of H, and BC the intensity of W, we have, from Art. 4, R = OC, the diagonal of the rectangle.

Otherwise, graphically, draw BD perpendicular to OC; then, if W, and H, be resolved, each into one component along OC and one at right angles to OC, we have

Components of 
$$H_1 = DO$$
 and  $BD$ ,  
Components of  $W_1 = CD$  and  $DB$ .  
 $DO + CD = R =$  pressure at  $DO = DD =$  o = rotatory effect.

If we suppose the rigid body extended so as to fill the space AOBC, then the resultant may be conceived as acting at any point in the line OC without altering its effect of translation on the whole mass. The effect within the body will, of course, be different for every new point of application. With this we are not now concerned.

We conclude, then, that if two forces whose lines of action are in the same plane act on a rigid body, and if from any point in the line of action of their resultant, perpendiculars be drawn to the lines of action of the forces, then the resistance at the point chosen, and the two given forces, will be in equilibrium, when the intensity and direction of the resistance are respectively equal and opposite to those of the resultant.

This conclusion may also be drawn from the figure, since two lines drawn from any point in OC, respectively perpendicular to the lines of action of  $W_1$  and  $H_2$ , must be proportional to l and h, and therefore equation (13) would be satisfied, whatever be the angle AOB.

If the two moments,  $W_rl$  and  $H_rh$ , are not equal, let us suppose that  $W_rl$  is the greater by reason of an increment given to  $W_r$ , so that l, h, and  $H_r$  remain unaltered. Then the resultant of the forces  $H_r$  and  $W_r$  will not pass through the point O, but will lie somewhere between it and the line of action of the augmented force  $W_r$ .

Suppose CE to be the line of action of the new resultant  $R_1$ , and draw OE perpendicular to CE; then will  $R_1 \times OE$  represent the total rotatory effect of the given pressures  $W_1$  and  $H_2$  with respect to the point O, and we shall have, if r = EO,

$$-W_{i}l + H_{i}h = -R_{i}r, \qquad (14)$$

where  $-R_1r$  is the moment of a couple, equivalent to the difference or algebraic sum of the moments of the couples whose arms are h and l.

We see, then, that the effect of one couple may be neutralized by the moment of another couple having the same axis of rotation and an opposite direction, and that the combined effort of two couples may be balanced by the moment of a single couple having the same centre of rotation.

11. The law may clearly be extended to any number of

forces,  $P_1$ ,  $P_2$ ,  $P_3$ , etc., acting in one plane to turn a rigid body about a fixed point in that plane, or about a fixed axis perpendicular to that plane. Let  $P_1$ ,  $P_2$ ,  $P_3$ , etc., be the lengths of the perpendiculars drawn from the fixed centre of rotation to the respective lines of action of  $P_1$ ,  $P_2$ ,  $P_3$ , etc. Let R be the resultant of translation of all the forces, and r the length of the perpendicular drawn from the same centre to the line of action of R; then

$$Rr = P_1 p_1 + P_2 p_2 + P_3 p_3 + \text{etc.} = \Sigma P p.$$
 (15)

The algebraic signs of the terms in this equation will depend upon the directions in which the forces tend to turn the rigid body; and it will be convenient to distinguish moments as positive which tend to turn the body in the direction taken by the hands of a watch, and to call moments having the opposite tendency negative.

For equilibrium we must have

$$\Sigma Pp = Rr = 0. \tag{16}$$

Equation (16) is satisfied either when R, the resultant of the given forces, becomes zero, or when r, the arm of the resultant couple, vanishes. In the former case the given impressed forces are in equilibrium among themselves; in the latter case, if R does not also vanish, it is equal and opposite to the resistance offered at the fixed point.

As in the case of two forces, each acting tangentially at one extremity of its lever arm to cause rotation about the fixed point common to the other extremities, so in the case of many forces acting in one plane on a rigid body, and tending to turn it about a fixed point, the resistance developed at the fixed point by each of the given forces will be equal and opposite to the given force, and will have its line of action parallel to that of the given force.

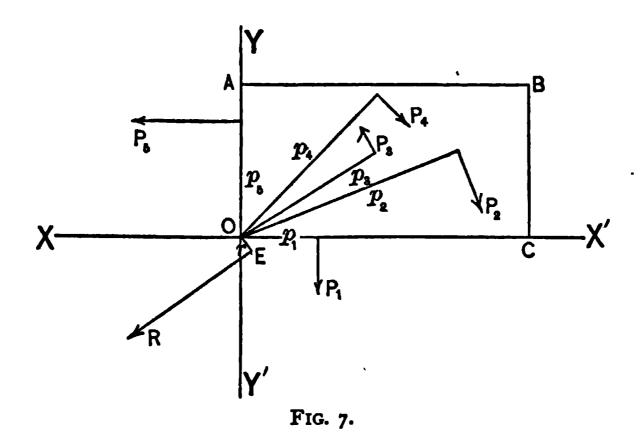
Hence the intensity and direction of the resultant, R, may be found from equations (4), (5), (8), and (9), as in the case of many forces acting in one plane at a common point.

12. And, having found R, equation (15) gives

$$r = \Sigma Pp + R. \tag{17}$$

If, then, through the fixed point a line be drawn parallel to the line of action of R, and through the same point another line be drawn at right angles to this line of action, and if on this second line the distance, r, be laid off from the fixed point, and R, both in magnitude and direction, be applied at the outer extremity of r, we shall have a graphical representation of the resultant couple whose moment is equivalent to the combined action of all the given forces.

13. The direction of R and the sign of Rr will show on which side of the fixed point r must be laid off.



EXAMPLE. — Let ABCO be a rigid body acted on by five forces, whose lines of action are in one plane, and which tend to turn the body about the fixed point O, Fig. 7.

Let the directions of the forces  $P_1$ ,  $P_2$ ,  $P_3$ , etc., and their points of application, be as indicated in the figure; and designate the angle between the line of action of any force and the axis of x by  $\alpha$ , and the distance of O from any point of application by p. Take

$$P_1 = 10 \text{ tons}, \quad p_1 = 4 \text{ feet}, \quad \alpha_1 = 270^{\circ} \text{ or } -90^{\circ}.$$
 $P_2 = 100 \text{ tons}, \quad p_2 = 12 \text{ feet}, \quad \alpha_2 = 290^{\circ} \text{ or } -70^{\circ}.$ 
 $P_3 = 20 \text{ tons}, \quad p_3 = 8 \text{ feet}, \quad \alpha_3 = 120^{\circ}.$ 
 $P_4 = 30 \text{ tons}, \quad p_4 = 10 \text{ feet}, \quad \alpha_4 = 315^{\circ} \text{ or } -45^{\circ}.$ 
 $P_5 = 200 \text{ tons}, \quad p_5 = 6 \text{ feet}, \quad \alpha_5 = 180^{\circ}.$ 

To find R, let these forces be considered as acting at the point O in direct opposition to the resistances there developed; then, by equations (4), (5), (8), and (9), we have

$$X = 10 \cos 270^{\circ} + 100 \cos 290^{\circ} + 20 \cos 120^{\circ} + 30 \cos 315^{\circ}$$

$$+ 200 \cos 180^{\circ} = -10 \cos 90^{\circ} + 100 \cos(-70^{\circ}) - 20 \cos 60^{\circ}$$

$$+ 30 \cos(-45^{\circ}) - 200 \cos 0^{\circ} = 10 \times 0 + 100 \times 0.34202$$

$$- 20 \times 0.5 + 30 \times 0.70711 - 200 \times 1 = 0 + 34.202 - 10$$

$$+ 21.2133 - 200 = -154.5847.$$

$$Y = 10 \sin 270^{\circ} + 100 \sin 290^{\circ} + 20 \sin 120^{\circ} + 30 \sin 315^{\circ} + 200 \sin 180^{\circ}$$

$$= 10 \times -1 + 100 \times -0.93969 + 20 \times 0.86603 + 30 \times -0.70711$$

$$+ 200 \times 0 = -10 - 93.969 + 17.3206 - 21.2133 + 0$$

$$= -107.8617.$$

$$R = \sqrt{(-154.5847)^{2} + (-107.8617)^{2}} = 188.496 \text{ tons.}$$

$$\tan \theta = \frac{-107.8617}{-154.5847} = 0.697753.$$

 $\theta = 34^{\circ} 54' 20''$ ; or, since X and Y are both negative, we must have

$$\theta = 214^{\circ} 54' 20''$$

and the resultant is therefore in the third quadrant.

From equation (15),

$$Rr = \Sigma Pp = +10 \times 4 + 100 \times 12 - 20 \times 8 + 30 \times 10 - 200 \times 6$$
  
= 40 + 1200 - 160 + 300 - 1200 = +180 foot-tons,

$$\therefore r = \frac{180}{188.496} = 0.95493 \text{ foot.}$$

Since the product Rr is positive, and the direction of the line of R, when drawn through the fixed point, is into the third quadrant, it follows that r must be laid off below the fixed point on the perpendicular to the line of R, as shown by OE in the figure; E being supposed rigidly connected with the solid AOCB.

## CHAPTER III.

MOMENTS OF THE EXTERNAL FORCES APPLIED TO A BEAM OR GIRDER.

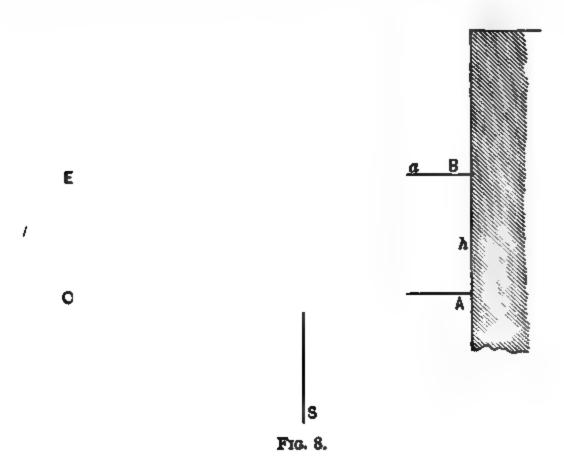
## SECTION I.

The Semi-Beam, or Girder fixed at One End and free at the Other.

14. We can now find expressions for the moments developed in any section of a beam or girder, by the action of any forces in the plane of the beam, in whatsoever manner applied

Let us first take a beam fixed at one end and free at the other, or a semi-beam as it is called. Let EOAB, Fig. 8, represent a beam fixed to a wall along the line AB = h. Suppose the weight of the beam to be w pounds for every unit of its length l = AO. Assume, also, that the length b = DC has an additional uniform load of w' pounds per linear unit, both w and w' being continuously distributed throughout their respective lengths. Also let W be a concentrated weight or pressure at the distance a' = BJ from the fixed end of the beam. Let BC = a = the distance from the wall to the nearer end of the uniform load w'b. Let P be any pressure acting at any point, G, with any inclination,  $\alpha$ , to the arm FG; and call the horizontal distance of the point G from the wall a'' = AK.

Suppose the beam horizontal, and all the applied pressures, except P, vertical. Let VS be any vertical section of the beam at the distance x from the fixed end AB. It is required to find the moment of the applied forces which must be resisted by the internal forces of the beam at the section VS.



Manifestly only the pressures at the left of VS affect that section. Taking the moments of these sinister pressures about any point, F, in the vertical section VS, and remembering that downward pressures on the left of VS give negative moments, we have the following equations for the required moment M:—

DEMINITURE LENGTH - 1. ICC TIE. U.	SEMI-GIRDER.	LENGTH	$\doteq I$ .	(See Fig.	8.)	)
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Load.	Conditions.	Force left of VS.	Arm.	Moments about F.	
W	x < a'	W	a' — x	M=-W(a'-x).	(18)
W	x = or > a'	0		M=0.	(19)
W	x = 0	W	a,	M = -Wa'.	(20)
W	a'=l	W	l-x	M=-W(l-x).	(21)
W	x = 0, a' = l	W	I	M = -Wl (max.).	(22)
wl	x < l	w(l-x)	$\tfrac{1}{2}(l-x)$	$M=-\tfrac{1}{2}w(l-x)^2.$	(23)
wl	x = l	0		M=0.	(24)
wl	x = 0	wl	$\frac{1}{2}I$	$M = -\frac{1}{2}wl^2 \text{ (max.)}.$	(25)
wb	x>a and $<(a+b)$	w'(a+b-x)	$\frac{1}{2}(a+b-x)$	$M = -\frac{1}{2}w'(a + b - x)^{2}.$	(26)
w'b	a = 0	w'(b-x)	$\frac{1}{2}(\delta - x)$	$M=-\frac{1}{2}w'(b-x)^2.$	(27)
wb	a=0, b=l	w'(l-x)	$\frac{1}{2}(l-x)$	$M=-\frac{1}{2}w'(l-x)^2.$	(28)
w'b	x = or < a	w'ò	$\frac{1}{2}b+a-x$	$M = -w'b\left(\frac{1}{2}b + a - x\right).$	(29)
w'b	x = 0, a = 0, b = l	w'l	<u>\</u> 2	$M = -\frac{1}{2}w'l^2 \text{ (max.)}.$	(30)
P	x < a''	P sin a	p	$M=P\sin ap.$	(31)

25. In applying these formulæ for W to the case of many equal weights placed at equal intervals along the beam, we may simplify the numerical computations by first summing the series resulting from assigning to a' and x their proper values.

Suppose we have n weights, each equal to W, at intervals of  $\frac{l}{n}$  feet along the beam; then

The moment at the fixed end of the beam, or when x = 0, due to all of the equal weights is, by summing (20),

$$M = -W\Sigma a' = -W!\left(\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots \frac{n}{n}\right)$$

$$= -W!\frac{n+1}{2}.$$
(32)

The moment at the fixed end due to  $1, 2, 3, \ldots r$ , of these equal weights, first in order is

$$M = -W\Sigma a' = -W! \left( \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots \frac{r}{n} \right)$$

$$= -W! \frac{r(r+1)}{2n}.$$
(33)

The moment at the fixed end due to the remaining (n - r) of these equal weights is

$$M = -W \sum d' = -W! \left( \frac{r+1}{n} + \frac{r+2}{n} + \frac{r+3}{n} + \dots \frac{n}{n} \right)$$

$$= -W! \left( \frac{n+1}{2} - \frac{r(r+1)}{2n} \right).$$
(34)

The moment at the interval r due to these remaining (n-r) equal weights is

$$M = -W \Sigma d' = -W \left( \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots \frac{n-r}{n} \right)$$

$$= -W \left( \frac{(n-r+1)(n-r)}{2n} \right)$$
(35)

Example 1. — A semi-girder projects 50 feet, and is loaded at intervals of 10 feet with a weight of 10 tons; required the moment at the fixed end due to the 5 equal weights.

From (32),

$$M = -W^{\frac{n+1}{2}} = -10 \times 50 \times \frac{6}{2} = -1500$$
 foot-tons.

Example 2. — The same conditions continuing, required the moment at the fixed end due to the first 3 of the weights.

From (33),

$$M = -W7 \frac{r(r+1)}{2n} = -10 \times 50 \times \frac{3(3+1)}{2 \times 5} = -600$$
 foot-tons.

EXAMPLE 3. — With same conditions of beam and load, required the moment due, at the fixed end, from the remaining 2 weights.

From (34),

$$M = -W(\frac{5+1}{2} - \frac{3(3+1)}{2 \times 5}) = -900$$
 foot-tons.

Example 4.—At 10 feet from the fixed end of the beam, what is the moment due to the 4 weights beyond?

From (35),

$$M = -W7 \frac{(5-1+1)(5-1)}{2 \times 5} = -1000$$
 foot-tons.

Example 5. — If the given semi-beam weighs 0.8 ton to the linear foot, what is the moment at its centre and at its fixed end?

From (23), if  $x = \frac{1}{2}l$ ,

$$M = -\frac{1}{2}w(\frac{1}{2}l)^2 = \frac{1}{8} \times 0.8 \times 50^2 = -250$$
 foot-tons.

From (25),

$$M = -\frac{1}{2} \times 0.8 \times 50^2 = -1000$$
 foot-tons.

Example 6. — Suppose the same beam to be covered with the uniform load 0.6 ton for the space of 15 feet, beginning 25 feet from the fixed end; required the moment due to this load at 30 feet from the fixed end.

Here

$$w' = 0.6, \quad b = 15, \quad a = 25, \quad x = 30.$$

From (26),

$$M = -\frac{1}{2} \times 0.6(25 + 15 - 30)^2 = -30$$
 foot-tons.

EXAMPLE 7. — If the load 0.6 ton per foot covers the first 35 feet of the beam, and the moment at 10 feet is required, we have b = 35, a = 0, x = 10; and, from (27),

$$M = -\frac{1}{2} \times 0.6(35 - 10)^2 = -187.5$$
 foot-tons.

EXAMPLE 8. — If the load 0.6 ton per foot covers the entire beam, the moment at the centre is, from (28),

$$M = -\frac{1}{2} \times 0.6 \times (50 - 25)^2 = -187.5$$
 foot-tons,

and at the fixed end

$$M = -\frac{1}{2} \times 0.6 \times 50^2 = -750$$
 foot-tons.

EXAMPLE 9. — If the uniform load 0.6 ton covers 40 feet of the beam, beginning at the free end, then the moment at 5 feet from the fixed end is, from (29),

$$M = -0.6 \times 40(\frac{1}{2} \times 40 + 10 - 5) = -600$$
 foot-tons.

Example 10. — If the force P, Fig. 8, is 4 tons, and its line of action makes an angle of 30° with the line GF = P = 20 feet, then the moment due to P at the point F is, from (31),

$$M = 4 \times 0.5 \times 20 = 40$$
 foot-tons.

16. If there are several concentrated weights,  $W_1$ ,  $W_2$ ,  $W_3$ , etc., or pressures,  $P_1$ ,  $P_2$ ,  $P_3$ , etc., or detached uniform loads,  $b_1w_1'$ ,  $b_2w_2'$ ,  $b_3w_3'$ , etc., at different points on the left of the section VS, we must evidently sum the moments due to the separate pressures for the total moment.

Thus we may write

$$M_x = -\sum W(a'-x) - \frac{1}{2}w(l-x)^2 - \frac{1}{2}w'(a+b-x)^2 - \sum w'(\frac{1}{2}b + a - x)b + \sum P\sin\alpha p, \quad (36)$$

where  $M_x$  is the moment, with reference to any point of any vertical section of a semi-beam, due to all the forces applied to the beam between its free end and the given vertical section.

It should be observed, that, for all pressures whose lines of action are vertical, the moments will be the same, whatever point of reference is taken in the vertical section VS; for such pressures have no horizontal component.

## SECTION 2.

17. We next take a beam or girder, horizontal, supported at its ends, and loaded in any manner whatsoever. Such a girder is also said to have its ends *free*; since they simply rest upon two level supports, and are fixed in no other manner.

## Fig. 9.

Let the beam OABE, Fig. 9, be supported at the two points. O and A, and be subjected to the following pressures:—

w = weight of beam per linear unit.

w' = uniform load per linear unit of the length b = CD.

W = a concentrated weight or vertical pressure at any point, K.

P = any force at distance a'' = OF from O.

 $V_{\rm r}$  = vertical re-action of the left support.

 $V_2$  = vertical re-action of the right support.

J = length of girder.

h = height of girder.

a = OD = the horizontal distance from the origin O to the nearer end of the uniform load bw'.

a' = the horizontal distance from O to the weight W.

x = the horizontal distance of any vertical section, VS, from O, the origin of co-ordinates.

y = the vertical distance of any point in the section VS from the horizontal line AO.

The vertical section VS is made by any plane cutting the beam perpendicular to the line AO.

It is required to find the moment generated at any vertical section, VS, by the action of each of the given pressures.

Since at any given cross-section, there can be but *one* moment due to the given simultaneous pressures, it follows that we may determine this moment, either by using the pressures applied upon the left side of the given section, or by using the applied pressures on the right side of the same section.

In the following table we use the pressures that act on the *left* of the section VS; and consequently downward pressures give negative moments, and upward pressures give positive moments, in accordance with our previous notation.

18. The sum of the re-actions  $V_1$  and  $V_2$  for the simple girder with free ends is equal to the total weight of the girder and its load.

The resistances  $V_1$  and  $V_2$  due to any concentrated weight, W, are, since there can be but one moment for the vertical section through W, inversely proportional to the horizontal distances of W from the points of support; and we have, from equation (13),

$$M = V_1 a' = V_2 (l - a'),$$

$$\therefore V_1 = V_2 \frac{l - d'}{d'} = W - V_2, \qquad (37)$$

$$\therefore V_2 = W \frac{a'}{l}. \tag{38}$$

$$V_i = W^{l-a'}. \tag{39}$$

Or, by proportion,

$$V_1: V_2:: l-a': a',$$

$$\therefore V_1 + V_2 : V_1 :: l : l - d', \quad \therefore \quad V_1 = W \frac{l - d'}{l}.$$

$$V_1 + V_2 : V_2 :: l : a', \qquad \therefore \quad V_2 = W \frac{a'}{l}.$$

Similarly, for the uniform load bw', the re-actions  $V_1$  and  $V_2$  will be inversely proportional to the distances of the centre of gravity of the uniform load from the points of support.

19. In the following table we have,—

First column, load whose moment is sought.

Second column, re-action at left support, giving +M.

Third column, conditions of load and plane VS.

Fourth column, part of load on left of VS, giving -M.

Fifth column, arm of  $V_{i}$ .

Sixth column, arm of load on left of VS.

BEAM SUPPORTED AT BOTH ENDS. MOMENTS AT ANY SECTION.

BEAM SUPPORTED AT BOTH ENDS.	E	1	MOMENTS AT ANY SECTION.	- 1
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	r.	Lond.		
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<i>p</i> − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 − 1 −	۰		, , , , , , , , , , , , , , , , , , ,	3
$\frac{a}{b} = \frac{1}{a} M$	ß		$M = W^{\frac{1}{2} - K'} \varphi'.$	3
2 - 1 A	4	A - 6	$M = W \frac{1 - d'}{t} z - W(x - a') = W \frac{1 - x'}{t} a'$	3
W-1-4	~	1-4	W=0.	3
x x x	* %		$M = \frac{M}{2} M^2 x$ , $M = \frac{M}{2} M^2$ (max.).	33
'sin a L	ħ	0	**	3
'sise.	ų	1 m	$M = -P \sin a \frac{l - a''}{r} x + P \sin a (x - a'')$ .	3
X max	h	18.sr		\$
X	۰ -	77	6	8
Xex	- 2 <del>4</del>	Ė	$M = \chi_{\rm col}^{12} - \chi_{\rm col}^{2} = \chi_{\rm col}^{2} ({\rm max.}),$	33
	4		$M = ev/\delta \frac{\delta - d - 3/\delta \pi}{2}.$	(\$3)
1-a-1	4		$N=\omega/b\frac{1-a-5b^2a}{a}$	3
1-a-1	14	K(x-e)	$H = w'b \frac{l-a-34b}{l} \pi - 54w'(x-a)^2$ .	3
$\frac{1}{4}$	4	35.85	N = w/b ! - 36 x - 36 w/ 54.	3
1-4-1	Ħ	5-0-36	$H = w/b \frac{L a - Xb}{l} x - w/b (x - a - Xb) = w/b (a + Xb) \frac{L x}{l} $ (57)	Es
X = 1	H	36 (x - a)	$K = \frac{\lambda^{3} x}{l} - \frac{\lambda^{3} $	₩,
7,0%	×	×	M = Xw/2 - Xw/12 = Xw/10 (max.).	3

20. Moments due Uniform Discontinuous Load on any Part of the Beam. — Let  $r_1 - r_2$  denote the number of equal weights, W, at equal intervals, c, between any two consecutive weights on the whole or any part of the girder. We may shorten the numerical computation of moments, as in case of the semi-girder, by first summing the series that arises in the expression for M.

For this purpose let  $r - r_2$  = the number of equal weights, W, on the length x;  $(r_2 + 1)c$  = the distance from the left end of the beam to the nearest weight. If this distance is less than c, that is, not a full interval, it follows that  $r_2$  will be a negative proper fraction. Now  $(r_1 - r_2) - (r - r_2) = r_1 - r$  = the number of equal weights between the point x and the right end of the beam. The three differences,  $r_1 - r_2$ ,  $r - r_2$ , and  $r_1 - r$ , must be integers, since each denotes a number of equal weights. If one of the three quantities r,  $r_1$ ,  $r_2$ , is not an integer, neither of the other two is an integer, and the decimal part of each is the same, except that, when  $r_2$  is negative, its value is less by unity than the common decimal part of r and  $r_1$ .

Let us first find the moment due  $r-r_2$  equal weights, W, at any point, x, between the last weight and right-hand end of the girder. We use equation (43), giving to a' the successive values  $c(r_2 + 1)$ ,  $c(r_2 + 2)$ ,  $c(r_2 + 3)$ , ... cr, and taking the sum; thus,

$$\Sigma a' = c[(r_2 + 1) + (r_2 + 2) + (r_2 + 3) + \dots r]$$

$$= \frac{1}{2}c(r - r_2)(r + r_2 + 1),$$

$$\therefore M_x = \frac{Wc}{2l}(r - r_2)(r + r_2 + 1)(l - x), \qquad (60)$$

where x cannot be less than cr.

Example 1.— Length of beam = l = 100 feet = 10c,  $r = 6\frac{1}{2}$ ,  $r_2 = 2\frac{1}{2}$ ; what is the moment at the fourth weight, W = 8 tons, due the 4 weights = 32 tons?

Here x = rc = 65,

... 
$$M_r = \frac{8 \times 10}{2 \times 100} \times 4 \times 10 \times 35 = 16 \times 35 = 560$$
 foot-tons.

If 
$$x = (r + 1)c = 75$$
,

$$\therefore M_{r+1} = 16 \times 25 = 400 \text{ foot-tons.}$$

If 
$$x = (r + 2)c = 85$$
,

... 
$$M_{r+2} = 16 \times 15 = 240$$
 foot-tons.

If 
$$x = (r + 3)c = 95$$
,

$$\therefore M_{r+3} = 16 \times 5 = 80 \text{ foot-tons.}$$

This shows a uniform decrease of moment for each interval beyond the given load.

Equation (40) gives for a single weight, W, applied at any point, a', the moment at any distance, x, between the weight and the left end of the beam. By giving to a' the successive values c(r + 1), c(r + 2), c(r + 3), ... cr, and summing for a' and a', we find

$$\Sigma a'^{\circ} = r_{1} - r,$$

$$\Sigma a' = c[(r+1) + (r+2) + (r+3) + \dots r_{1}] = \frac{1}{2}c(r_{1} - r)(r_{1} + r + 1).$$

$$M_{x} = \frac{W}{2l}[2(r_{1} - r)l - c(r_{1} - r)(r_{1} + r + 1)]x, \quad (61)$$

which is the moment at any point, x, between the left end of the girder and the nearest weight, which is at the  $(r + 1)^{th}$  point of division; the number of weights being  $r_1 - r_2$ , and x not being greater than c(r + 1).

EXAMPLE 2.—Let l = 100 feet = 10c, W = 8 tons,  $r_1 = 6\frac{1}{2}$ ,  $r = 2\frac{1}{2}$ ; what is the moment at the first weight due the  $r_1 - r_2 = 4$  equal weights?

Here 
$$x = (r + 1)c = 35$$
,

... 
$$M_{r+1} = \frac{8}{2 \times 100} (2 \times 4 \times 100 - 10 \times 4 \times 10) 35 = 560$$
 foot-to

$$x=rc=25$$

$$M_r = 16 \times 25 = 400$$
 foot-tons.

$$x=(r-1)c=15,$$

... 
$$M_{r-1} = 16 \times 15 = 240$$
 foot-tons.

$$x=(r-2)c=5,$$

... 
$$M_{r-2} = 16 \times 5 = 80$$
 foot-tons.

This shows a uniform decrease of moment for each interval before the given load. These moments are the same as those of example I, as they should be, since the same load is symmetrically placed on the beam in both cases.

If we add equations (60) and (61), calling the sum  $M_x$  still, we shall have

$$M_x = \frac{W}{2l} \Big\{ [2(r_1 - r)l - c(r_1 - r_2)(r_1 + r_2 + 1)] x + cl(r - r_2)(r + r_2 + 1) \Big\}, \quad (62)$$

which is the moment at any point, x, of the beam due  $r_1 - r_2$  equal weights, W, placed at equal and consecutive intervals, c, over the whole or any part of its length.

Here r cannot be less than  $r_2$  nor greater than  $r_1$ , and x lies between rc and (r + 1)c for the loaded part of the beam, but may have any value between 0 and  $r_2c$  where  $r = r_2$ , and any value between  $r_1c$  and l where  $r = r_1$ .

EXAMPLE 3. — Let l = 100 feet = 10 c, W = 8 tons,  $r_2 = 2\frac{1}{2}$ ,  $r_1 = 6\frac{1}{2}$ ; what is the moment at the fourth weight? Here  $x = r_1c = 65$  feet, and (62) gives, If  $r = r_1$ ,

$$M_{r_1} = \frac{8}{200} \Big\{ (0 - 10 \times 4 \times 10) 65 + 10 \times 100 \times 4 \times 10 \Big\} = 560$$
 foot-tons.  
Or, if  $r = r_1 - 1$ ,

$$M_{r_1} = \frac{8}{200} \left\{ (2 \times 1 \times 100 - 10 \times 4 \times 10) 65 + 10 \times 100 \times 3 \times 9 \right\}$$
  
= 560 foot-tons.

What is the moment one interval beyond the last weight? Here  $x = (r_1 + 1)c = 75$ , and  $r = r_1 = 6\frac{1}{2}$ , in (62);

... 
$$M_{r_{1}+1} = \frac{8}{200} \{ (0 - 10 \times 4 \times 10) 75 + 10 \times 100 \times 4 \times 10 \}$$
  
= 400 foot-tons.

If n = the whole number of intervals in the girder's length, we have  $c = \frac{l}{n}$ , and (62) becomes

$$M_{x} = \frac{W?}{2n} \Big\{ [2n(r_{1} - r) - (r_{1} - r_{2})(r_{1} + r_{2} + 1)] \frac{x}{\ell} + (r - r_{2})(r + r_{2} + 1) \Big\}, \quad (63)$$

from which we may find the simultaneous moments at all points throughout the girder due to any uniform discontinuous partial or full load. EXAMPLE 4.—Let a uniform load of 4 weights, each = W = 8 tons, spaced c = 10 feet from weight to weight, come upon a girder 100 feet long, and move forward to the centre; required the simultaneous moments throughout the girder as the foremost end of the load passes the points x = 5, 15, 25, 35, etc., n = 10.

Owing to the important applications of this formula which are to follow, we add the complete solution of this problem, and may remark that by giving to x the values 10, 20, 30, 40, etc., we can find the simultaneous moments at the full intervals as the foremost end of the load passes the successive points of division. Or we may give x any value we please between 0 and l, and so suit the equal or unequal panel lengths of any girder. As the load, now central, passes off to the right, it is evident these moments will be reversed.

LOAD.
UNIFORM
ADVANCING
DOE
MOMENTS DUE
SIMULTANEOUS

$M_{10}$	<b>(4)</b>	<b>∞</b>	ı	<b>%</b>		33	)		•	<b>\$</b>				3	\$				•	8	
M <sub>o</sub>	9	24		ጟ		8				144					*					240	
M,	Io	\$		8		8				240					5			•		<u>8</u>	
M,	14	\$6	,	126		224				336				0,	<b>}</b>				•	8	
M,	18	72		162		288				432	-			7-1	2/2	_		•	<u>호</u>		
M's	22	88		861		352	}			528				024				<u>Ş</u>		,	9
W.	<b>5</b> 2	104		234		416	•		24	)			592			7	8	_	•	-	592
M,	30	120		270		8	•	28,	}		(	480			,	8				(	<del>8</del>
M,	8	136	226		ठू		,	336			88					240		_		•	336
M,	38	72	102	128			112				8				0	8				(	128
Equation (63).	( 0 - 12)0.0	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$(20 \times 2 - 4^2)0.1$	(20 × 1 — 4°)0.2 (0 — 4°)0.3	$(20 \times 4 - 4 \times 6)0.05 +$	$(20 \times 3 - 24)$ 0.15 + 1 ×	8 8	(o -24)0.45 + 4 ×	$(20 \times 4 - 4 \times 8)0.05 +$	$(20 \times 3 - 32)0.25 + 1 \times$	$(20 \times 2 - 32)0.35 + 2 \times$	$(20 \times 1 - 32)0.45 + 3 \times$	( 0 — 32 )0.55 ± 4 ×	$(20 \times 4 - 4 \times 10) \text{ o.c5} +$	(20 × 3 — 40 )0.35 + I ×	$(20 \times 2 - 40)0.45 + 2 \times$	$(20 \times 1 - 40)0.55 + 3 \times$	( o — 40 )0.65 + 4 ×	•
<u> </u>	8	<del>\$</del> <del>\$</del>	<b>4 4</b>	<del>4</del> <del>6</del>	<u> </u>	<del>4</del> <del>4</del>	4	<del>\$</del>	<del>4</del> <del>4</del>	\$	6	4	<u>수</u>	6	<del>수</del>	<del></del>	<u>.</u> 6	<del>6</del>	<del>4</del>	<del>0</del>	•
<b>41~</b>	0.05	0.05	0.05	0.25	0.15	0.25	0.05	0.15	0.25	0.45	0.05	0.25	0.35	0.45	0.55	Q (	0.35	0.45	0.55	0.65	ments
5	0.5	1.5	2. 5. 5. 5.	2.5		3.5 7.5	4.5	4.5	4 4 3 7	4.5	5.5	5.5	5.5	5.5	ج ک	5.5		6.5	6.5	6.5	mo
<b>.</b>	0.5	0.5	0.5		1.5	2.5	0.5	1.5	3. S.	4.5	1.5	2.5	3.5	4.5	5.5	2.5	3.5	4.5	5.5		ma
£.	-0.5	10.5	10.5	10.5	-0.5	0.5	+0.5	+0.5	+ + 0, 0,	Ö	+1.5	+1.5	÷.	i i	Ξ,	+2.5	ri	+2.5	+2.5	+2.5	Maxi
No. of Wts.	-	н н	ကက	m <b>-</b>	+ 4	4 4	4	4	4 4	4	4	4	4	4	4	4	4	4	4	4	

In the above table all moments included by the same brace are simultaneous, and due, at their respective points, to all the weights on the girder, as indicated by the first column,  $r_1 - r_2$ .

Only the first moment in any horizontal line is computed by the formula in that line; the remaining moments in any line being found by the simple variation of x, using only the term containing x. In this example the constant difference to be added to the first moment in any horizontal line of moments is four times the quantity in the parenthesis for the given line.

21. Let the length, *l*, of the girder be divided into *n* equal intervals, c, so that there are n-1 points of division; then, if a weight, W, be applied at each point of division beginning at the left, we may find the moment at the foremost end of this advancing load, from equation (60), by putting  $r_2 = 0$ ,  $x = rc = \frac{rl}{r}$ Thus,

$$M_r = \frac{W}{2n}r(r+1)(l-cr) = \frac{Wl}{2n^2}r(r+1)(n-r),$$
 (64)

which is the moment at the foremost end of a uniform discontinuous load when that end passes the  $r^{th}$  point of division, and r equal weights have come on.

Example. — Let l = 100 feet, n = 10, W = 8 tons; what is the moment at each point of division as the foremost end of this load passes it? Using (64),

```
If r = 1, M_1 = 1 \times 2 \times 9 \times 4 = 72 foot-tons.
         2, M_2 = 2 \times 3 \times 8 \times 4 = 192 foot-tons.
         3, M_3 = 3 \times 4 \times 7 \times 4 = 336 foot-tons.
         4, M_4 = 4 \times 5 \times 6 \times 4 = 480 foot-tons.
         5, M_5 = 5 \times 6 \times 5 \times 4 = 600 foot-tons.
         6, M_6 = 6 \times 7 \times 4 \times 4 = 672 foot-tons.
         7, M_7 = 7 \times 8 \times 3 \times 4 = 672 foot-tons.
         8, M_8 = 8 \times 9 \times 2 \times 4 = 576 foot-tons.
         9, M_9 = 9 \times 10 \times 1 \times 4 = 360 foot-tons.
```

22. From equation (63), by putting  $r_2 = 0$ ,  $r_1 = n - 1$ , and  $x = rc = \frac{rl}{n}$ , we derive

$$M_r = \frac{W!}{2n}(n-r)r, \qquad (65)$$

which is the moment at the  $r^{th}$  weight due n-1 equal weights, W, placed at equal intervals,  $\frac{l}{n}$ , throughout the girder.

Example. — Uniform discontinuous load; W = 8 tons, l = 100 feet, n = 10.

If 
$$r = 1$$
,  $M_1 = 40 \times 9 \times 1 = 360$  foot-tons.  
2,  $M_2 = 40 \times 8 \times 2 = 640$  foot-tons.  
3,  $M_3 = 40 \times 7 \times 3 = 840$  foot-tons.  
4,  $M_4 = 40 \times 6 \times 4 = 960$  foot-tons.  
5,  $M_5 = 40 \times 5 \times 5 = 1000$  foot-tons.

And these moments are to be reversed for the other half-span.

23. Suppose that the first and last intervals into which the beam is divided are each  $=\frac{1}{2}c=\frac{l}{2n}$ , while every other is =c, and that we wish to find the moment at the foremost end of a uniform load of equal intervals, c, as that end passes each point of division of the beam.

For this object we employ equation (60), making  $x = rc = \frac{rl}{n}$ , and  $r_2 = -\frac{1}{2}$ , and have

$$M_r = \frac{W}{2n}(r + \frac{1}{2})^2(l - rc) = \frac{W?}{2n^2}(r + \frac{1}{2})^2(n - r).$$
 (66)

Example. — Let l = 100 feet, W = 8 tons, n = 10.

For 
$$r = 0.5$$
,  $M_1 = 4 \times 1 \times 9\frac{1}{2} = 38$  foot-tons.  
1.5,  $M_2 = 4 \times 4 \times 8\frac{1}{2} = 136$  foot-tons.  
2.5,  $M_3 = 4 \times 9 \times 7\frac{1}{2} = 270$  foot-tons.  
3.5,  $M_4 = 4 \times 16 \times 6\frac{1}{2} = 416$  foot-tons.  
4.5,  $M_5 = 4 \times 25 \times 5\frac{1}{2} = 550$  foot-tons.  
5.5,  $M_6 = 4 \times 36 \times 4\frac{1}{2} = 648$  foot-tons.  
6.5,  $M_7 = 4 \times 49 \times 3\frac{1}{2} = 686$  foot-tons.  
7.5,  $M_8 = 4 \times 64 \times 2\frac{1}{2} = 640$  foot-tons.  
8.5,  $M_9 = 4 \times 81 \times 1\frac{1}{2} = 486$  foot-tons.  
9.5,  $M_{10} = 4 \times 100 \times \frac{1}{2} = 200$  foot-tons.

24. From equation (63), by making  $r_2 = -\frac{1}{2}$ ,  $r_1 = n - \frac{1}{2}$ , and  $x = \frac{rl}{n}$ , we also obtain

$$M_r = \frac{W!}{8n} \Big\{ 4r(n-r) + 1 \Big\}, \tag{67}$$

which is the moment at any weight due n equal weights, W, applied at equal intervals,  $\frac{l}{n}$ , except the interval at each end, which is  $=\frac{l}{2n}$ . Here r takes the values  $\frac{1}{2}$ ,  $\frac{3}{2}$ ,  $\frac{5}{2}$ ,  $\frac{7}{2}$ , ...  $\frac{2n-1}{2}$ , and n denotes the whole number of full intervals in the length, l, which in this case is also the whole number of weights.

Example. — Let, as before, l = 100 feet, n = 10, W = 8 tons; then

For 
$$r = 0.5$$
,  $M_1 = 10(1 \times 19 + 1) = 200$  foot-tons.  
1.5,  $M_2 = 10(3 \times 17 + 1) = 520$  foot-tons.  
2.5,  $M_3 = 10(5 \times 15 + 1) = 760$  foot-tons.  
3.5,  $M_4 = 10(7 \times 13 + 1) = 920$  foot-tons.  
4.5,  $M_5 = 10(9 \times 11 + 1) = 1000$  foot-tons.

The same to be reversed for the other half.

25. Difference of Simultaneous Moments at Consecutive Points of Division. — By making  $r_2 = 0$ , and x = (r + 1)c, in equation (60), we have

$$M_{r+1} = \frac{W}{2n}r(r+1)[l-(r+1)c]$$

$$= \frac{Wl}{2n^2}r(r+1)(n-r-1), \quad (68)$$

which is the moment one interval, c, beyond the foremost end of a uniform load consisting of r equal weights, W, at the distances c, 2c, 3c, ... rc, respectively, from the left end of the beam.

Subtracting (64) from (68), we have

$$\Delta M = M_{r+1} - M_r = -\frac{Wc}{2n}r(r+1), \qquad (69)$$

which is the difference between the simultaneous moments at any two consecutive points of division on the unloaded end of a beam, which has r equal weights at full intervals on the other end.

For finding the difference of simultaneous moments at consecutive points of division on the loaded part of the beam, we use (62), making  $r_2 = 0$ , and x = rc, (r + 1)c, (r + 2)c, etc., in succession. Thus,

$$M_{r} = \frac{W}{2l} \Big\{ [2(r_{1}-r)l - cr_{1}(r_{1}+1)]rc + clr(r+1) \Big\},$$

$$M_{r+1} = \frac{W}{2l} \Big\{ [2(r_{1}-r)l - cr_{1}(r_{1}+1)](r+1)c + clr(r+1) \Big\},$$

$$+ clr(r+1) \Big\},$$

$$\Delta M = M_{r+1} - M_{r} = \frac{Wc}{2n} \Big\{ 2(r_{1}-r)n - r_{1}(r_{1}+1) \Big\}, \quad (70)$$

which is the first order of differences for the loaded part or end of the beam: and  $\Delta M$  is an increasing function of  $r_1$  for a given value of  $r_2$  and will be greatest when  $r_2$  is greatest;

that is, when  $r_1 = n - 1$ , its limit. But at this limit of  $r_1$  the girder is loaded, and the positive differences on the left half will equal the negative differences for the corresponding intervals on the right half of the girder. Putting n - 1 for  $r_1$  in (70), we have

$$\Delta M_r = \frac{Wc}{2} \Big\{ n - 1 - 2r \Big\}, \tag{71}$$

which gives the difference of simultaneous moments for each interval, c, of the beam fully loaded with n-1 weights, W, applied at equal and all full intervals, c, or with n weights, W, when each end interval  $= \frac{1}{2}c$ .

Subtract (69), which is negative, from (71), whose positive values in one half-span are equal to its corresponding negative values in the other half-span, and the remainder is

$$\frac{Wc}{2n}\Big\{(n-r)^2-(n-r)\Big\},\,$$

which is positive, since n > r, and both n and r are integers.

Therefore the greatest negative difference computed by (71) for any interval is numerically less than the difference computed by (69) for the same interval in the second half-span; that is, both half-spans, if the uniform load is to travel either way. Consequently we use (69) in finding the greatest difference of simultaneous moments for any interval due a uniform discontinuous moving load.

It may be observed here that (69), for the unloaded end of the beam, gives a constant first difference, while (70), for the loaded end, gives a first difference which is not constant. By putting r + 1 for r in (70), and subtracting (70) from the resulting equation, we find the second difference,

$$\Delta(\Delta M) = -Wc, \qquad (72)$$

which is constant and negative, and may be conveniently employed in some computations.

Example 1.— Let a girder of 10 panels, each 10 feet, be laden with a permanent load of 4 tons at each panel point, and a discontinuous uniform rolling load of 8 tons to be applied at the same points as the load advances; required the greatest moments at these panel points, and the greatest difference of simultaneous moments at any two consecutive panel points, due to both these loads.

The greatest moments will occur when both loads cover the beam. We have, then, in equation (65), W = 12, l = 100, n = 10,  $\frac{Wl}{2n} = 60$ , and r = 1, 2, 3, etc., in succession, for the greatest moments.

The difference of moments at consecutive panel points due dead load is to be computed by (71), making W = 4, c = 10, n = 10,  $\frac{Wc}{2} = 20$ , and r = 0, 1, 2, 3, 4, etc., in succession.

And the greatest difference of simultaneous moments for each interval due live load is found by using equation (69), when W=8, c=10, n=10,  $\frac{Wc}{2n}=4$ , and r=1,2,3,4, etc., in succession.

COMPUTATION FOR GREATEST MOMENTS AND DIFFERENCES.

No. of the Panel Point, r.	0	1	2	8	4	5	6	7	8	9
Greatest moments										
=60(10-r)r	•	540	960	1260	1440	1500	1440	1260	960	540
Differences, dead load			i	_						
= 20(9 - 2 <b>r</b> )	180	140	100	60	20	-20	<b>—6</b> 0	-100	-140	—180
Greatest differences, live			1	[						ı
$\log d = -4r(r+1)$	0	8	-24	48	<b>8</b> 0	-120	<b>—168</b>	-224	<b>288</b>	<b>—360</b>
Total differences for both			ĺ							
loads	<b>18</b> 0	132	76	12	60	-140	228	324	<b>428</b>	-540
Differences, load moving (		•			60	-140	228	-324	-428	-540
either way	540	428	324	228	140	60			·	

If in equation (66), instead of the factor (l - rc), we write [l - (r + 1)c], and then subtract (66) from the resulting equation, we shall have

$$\Delta M = -\frac{Wc}{2n}(r + \frac{1}{2})^2, \tag{73}$$

which gives the greatest difference of simultaneous moments at any two consecutive points of division, due live load advancing by equal panel weights, when the two extreme panels have each but half the length of every intervening panel. Here observe that r takes the successive values  $-\frac{1}{2}$ ,  $\frac{3}{2}$ ,  $\frac{5}{2}$ , ...  $\frac{2n-1}{2}$ , and that  $c=\frac{l}{2n}$  for the two extreme panels, but  $c=\frac{l}{n}$  for all others.

EXAMPLE 2. — Given the same loads and length of girder as in example 1, but the panel points being now at the distances 5, 15, 25, 35, etc., from either end; required the greatest moment at each of these points, and the greatest difference of simultaneous moments at any two consecutive panel points, for both live and dead loads.

Equation (67) gives the greatest moments if we make l = 100, n = 10, W = 12,  $\frac{Wl}{8n} = 15$ , and  $r = \frac{1}{2}$ ,  $\frac{8}{2}$ ,  $\frac{5}{2}$ ,  $\frac{7}{2}$ , etc., in succession.

The differences for dead load are computed from (71) by putting W=4, c=5 in two-end panels, c=10 in all others, n=10, and  $r=-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2}$ , etc.,  $\frac{Wc}{2}=10$  or 20.

The greatest differences for live load are found from (73), where W=8, c=5 or 10, n=10, and  $r=-\frac{1}{2},\frac{3}{2},\frac{5}{2}$ , etc.,  $\frac{Wc}{2n}=2$  or 4.

Computation	FOR	GREATEST	Momen	NTS .	AND	GREATEST	SIMULTANEOUS
		DIFFERENCE	S FOR	EACH	INT	ERVAL.	

r.		ł	2	\$	<del>7</del>	9	Å	13	疮	17	ाँठ
Greatest moments, $\frac{W!}{8n} \left\{ 4r(n-r) + 1 \right\}$		300	<i>7</i> 80	1140	1380	1500	1500	1380	1140	780	300
Differences, dead load, $\frac{Wc}{2} \left( n - 1 - 2r \right)$ Differences, live load,	100	160	120	80	40	0	-40	80	<b>—120</b>	<b>—160</b>	—100
$-\frac{Wc}{2n}\left(r+\frac{1}{2}\right)^{2}$	0	-4	<b>—16</b>	<b>—</b> 36	64	—100	—z44	<b>—196</b>	256	<del>-324</del>	-200
Total differences	100	`156	704	44	-24	<b>—</b> 1∞	<b>—184</b>	<b>—27</b> 6	<b>—37</b> 6	-484	-300
Differences to be used	300	484	376	276	<del>-24</del> 184	—100 100	—184 24	-276	<b>—376</b>	<b>—484</b>	<del>-300</del>

26. To determine the Point in any Girder simply supported at its Two Ends, and carrying any Partial or Complete Uniform Discontinuous Load, where the Moment due that Load is Greatest. — The required greatest moment will occur at a point within the loaded part of the girder, since for any partial load the simultaneous moments decrease from either end of the load to the corresponding end of the girder.

If, therefore, we put  $x = \frac{rl}{n}$  in (63), and call the result  $M_r$ , then in  $M_r$  thus found put (r + 1) for r, giving  $M_{r+1}$ , and equate  $\Delta M_r = M_{r+1} - M_r$  to zero, we shall find

$$r = r_1 - \frac{(r_1 - r_2)(r_1 + r_2 + 1)}{2n};$$
 (74)

and the panel point of greatest moment lies between rc and (r + 1)c, except when rc and (r + 1)c are panel points.

Let us verify this statement by referring to example 4,

article 20. Taking  $r_1$ ,  $r_2$ ,  $r_3$ , and the greatest moment, from that example, we compute r by (74), and write as below:—

$r_{i}$ .	72.	r.	Mmax.	r by (74).
6.5	2.5	4.5 or 5.5	640	4.5
5.5	1.5	4.5	624	3.9
4.5	0.5	3.5	544	3.3
3.5	-0.5	<b>3·</b> 5	416	2.7
2.5	-0.5	2.5	270	2.05
1.5	<b>-0.5</b>	1.5	136	1.3
0.5	-o.5	0.5	38	0.45

27. If in equation (63) we make  $r = r_1$ ,  $x = \frac{r_1 l}{n}$ ,  $r_2 = r_1 - e$ ,

e being the number of equal weights on the beam, we shall find, after putting  $\Delta M_{r_1} = M_{r_{1+1}} - M_{r_1} = 0$ ,

$$r_{\scriptscriptstyle \rm I} = \frac{2n + e - 3}{4}.\tag{75}$$

But when the advancing load reaches back to the left end of the girder, we may not know how many weights will give a maximum moment at the foremost weight. In that case we deduce  $\Delta M_r = M_{r+1} - M_r = 0$  from (64), and find

$$r = r_1 = \frac{2}{8}(n - 1) \tag{76}$$

for a girder of equal panels to receive an advancing load of equal weights applied at successive panel points.

And for a girder each of whose two extreme panels is one-half of any other, the advancing load to be applied at panel points, we derive  $\Delta M_r = M_{r+1} - M_r = 0$ , from (66), and get

$$r = r_1 = \frac{1}{6}(2n - 5 \pm \sqrt{4n^2 + 4n - 2}). \tag{77}$$

In all these cases the panel point at foremost end, having the greatest moment as the load advances, lies between  $r_1c$  and  $(r_1 + 1)c$ , except when  $r_1c$  and  $(r_1 + 1)c$  are panel points.

<del> </del>	•				·		-								
M.	<b>∞</b>	7.7		84			72			8			120		
K.	16	84		8			144			192		_	240		
M <sub>1</sub>	24	72		144			216			288			360		
M.	32	8		192			288	_		384			480		
Ms	0	120		240			360			480		\$20		520	172
M,	84	144	<del> </del>	288			432		496		480	<del>,</del>		496	-48 +120
M,	\$6	168		336		424		432			360			432	-24 +144
M,	\$	192	304	1	336		-	288			24c			336	+168
M <sub>1</sub>	72	136	192		168			144			120			192	+192
From Equation (63).	( o -1 × 2)0.1 +1 × 2	$(20 \times 1 - 2 \times 3)0.1 + 1 \times 2$ (0 - 6)0.2 + 2 × 3	$(20 \times 2 - 3 \times 4)$ 0.1 + 1 × 2 $(20 \times 1 - 12)$ 0.2 + 2 × 3	- 12)0.3 + 3 ×	$(20 \times 3 - 3 \times 6)0.1 + 0 \qquad ($	$ (20 \times I - I8)0.3 + 2 \times 5 $	$(0 - 18)0.4 + 3 \times 6$	$(20 \times 3 - 3 \times 8)0.1 + 0$	$ (20 \times 1 - 24)0.4 + 2 \times 7 $	$(0 - 24)0.5 + 3 \times 8$	$(20 \times 3 - 3 \times 10)0.1 + 0$	$(20 \times I - 30)0.5 + 2 \times 9$	(o - 30)0.6+3×10	•	simultaneous moments {
2 × 2	9	<b>6 6</b>	<b>4 4</b>	\$	9	40	9	9	9	9	9	<b>\$</b>	9	•	simı
#1~	0.1	0.1	0.1	0.3	0.1	0.3	0.4	0.1	0.4	0.5	0.1	0.5	9.0	•	es of
2	H	n n	т т	m	4	4	4	2	S	2	9	9	9	moments	differences of
	Ħ	- 8	H 6	က	H	က	4	8	4	Ŋ	က	2	9		
ξ.	0	0 0	0 0	0	H	H	H	4	4	8	က	m	٣	Maxima	Maxima
\$ - L3	<b>—</b>	n n	m m	, w	က	3	က	m	က	n	3	3	က	Ä	K

We now give an example of an equal-panelled girder traversed by an odd number of equal apex weights, and will then illustrate the application of equations (75), (76), and (77).

EXAMPLE. — Girder of 10 panels 10 feet each; 3 weights, of 8 tons each, at intervals of 10 feet between consecutive weights. What are the simultaneous moments at all the panel points, as the foremost weight of this load passes each panel point in succession? Use equation (63), where, now, W = 8, l = 100, n = 10,  $r_2 = 0$ , 1, 2, 3, etc., in succession.

Moments within the same brace, simultaneous. (See table, p. 42.)

Formula gives only the first moment in any horizontal line of moments. For other moments in same line, add four times the parenthetic quantity to the moment immediately before the required moment.

For the greatest moment at foremost end of this moving load, we have from (75), where, now, e = 3,

$$r_1 = \frac{20 + 3 - 3}{4} = 5,$$

$$r_1 + 1 = 6,$$

which agrees with the above table; the moment being 480 when the foremost end of load passes either of these points.

Also, if e = 4, as in example 4, article 20, we have, from (75),  $r_1 = \frac{20 + 4 - 3}{4} = 5.25$ ; and 5.50, which gives the greatest moment 576, at foremost end of load, lies between 5.25 and 6.25.

For a full load coming upon the panel points of a girder having 10 equal panels, (76) gives  $r_1 = \frac{2}{3}(10 - 1) = 6$ ,  $r_1 + 1 = 7$ , a result in accord with the solution in article 21, where the moment at foremost end is greatest, and equals 672 at these two points.

Also, when n = 10, (77) gives  $r_1 = 5.98$ ,  $r_1 + 1 = 6.98$ ; and 6.5, giving 686 foot-tons (example of article 23), lies between 5.98 and 6.98.

28. To find the general expression for the point of greatest moment, on the loaded part of the beam, due to a uniform dead load, a weight, W, being applied at each of the (n-1) or n panel points, and to a uniform live load consisting of  $r_1 - r_2$  equal weights, L, applied at consecutive panel points as the load advances, we employ equation (63), putting L for W, and  $x = \frac{rl}{n} = rc$ , and so have  $M_r$ . Then, substituting r + 1 for r, we get  $M_{r+1}$ ,

for the loaded part of the beam.

This expression added to (71), and the sum made equal to 0, gives

$$r = \frac{L}{2n(L+W)} \left\{ n(n-1)\frac{W}{L} - (r_1 - r_2)(r_1 + r_2 + 1) + 2nr_1 \right\}$$
 (78)

to be used when the value of r lies between  $r_2$  and  $r_1$ ; and the panel point under the live load, having the greatest moment, lies between rc and (r + 1)c when rc and (r + 1)c are not panel points.

In a similar manner, putting  $r = r_2$ , and  $x = r_2c$ ,  $(r_2 - 1)c$ , in succession, in (63), finding  $\Delta M_{\tilde{r}_2}$  and adding it to  $\Delta M$  in (71), we derive

$$r = \frac{1}{2}(n-1) + \frac{L}{2nW}(r_1 - r_2)[2n - (r_1 + r_2 + 1)]$$
 (79)

where r is not greater than  $r_2$ , that is, at the left of a partial live load. Also, when the point of greatest moment, consider-

ing dead and live loads, lies beyond the live load, we derive, from (63),

$$r = \frac{1}{2}(n-1) - \frac{L}{2nW}(r_1 - r_2)(r_1 + r_2 + 1), \qquad (80)$$

where r is not less than  $r_1$ , that is, beyond the live load.

29. We next assume that the uniform load, w units of weight per linear unit of beam, advances by continuous increments, and not by leaps, or entire panel weights added at once; and require the moment at the foremost end of a load which is thus uniformly distributed continuously from its foremost end to the left end of the girder.

Equation (56) applies here if we make x = b = length of uniform load measured from the left support; and we have

$$M_{bw'} = \frac{w'b^2}{2l}(l-b).$$
 (81)

And, if w is the unit weight of the dead load, we have from (49), by putting x = b,

$$M_{bw} = \frac{1}{2}wb(l-b), \qquad (82)$$

where  $M_{bw}$  is the moment of a beam, at the distance b from one extremity, due to the unit weight, w, covering the entire beam.

For the total moment due to live and dead loads at the foremost end of bw', we take the sum of (81) and (82), and have

$$M_{bw'+bw} = \frac{\frac{1}{2}b(l-b)}{l}(w'b+lw).$$
 (83)

Example. — Let l = 100, w = 0.4 ton, w' = 0.8 ton; and find the moments at the foremost end of the moving load, bw', at intervals of 10 feet as it advances. From (83), —

д.	$\frac{1}{2}b\frac{l-b}{l}.$	w'b + lw.	Mbw + bw.
0	o	40	0
10	4.5	48	216
20	<b>8.</b> o	56	448
30	10.5	64	672
40	12.0	72	864
50	12.5	80	1000
60	12.0	88	1056
70	10.5	96	1008
80	8.0	104	832
90	4.5	112	504
100	0	120	0

Each of these moments is, as it should be, less than that found for apex loads by just the moment due  $\frac{1}{2}w'$  at the point taken, since the point at the end of the continuously distributed live load sustains but half a panel weight of the live load.

30. If it be required to find the moment due to both dead load, lw, and live load, bw', at any point ahead of the latter, we use for the live load (57) by making a = 0, and for the dead load (49), and have

$$M_x = \frac{w'}{2l}b^2(l-x) + \frac{1}{2}w(l-x)x. \tag{84}$$

Or, for (r + 1) intervals, each equal to  $\frac{l}{n}$ , we find, if  $b = \frac{rl}{n}$ , and  $x = \frac{(r + 1)l}{n}$ ,

$$M_{r+1} = \frac{w'l^2}{2n^3}r^2(n-r-1) + \frac{wl^2}{2n^2}(r+1)(n-r-1). \quad (85)$$

EXAMPLE. — Let l = 100, w' = 0.8, w = 0.4, n = 10; and find the total moment at the distance x = b + 10, or at the end of the (r + 1)<sup>th</sup> interval,  $\frac{l}{x}$ 

First computation, using (84),—

<i>δ</i> .	x.	<u>w'</u> .	<i>გ</i> 2.	l-x.	First Term.	¾w.	l-x.	x.	Second Term.	M <sub>X</sub> .
0	10	0.004	0	90	0	0.2	. 90	10	180	180
10	20	0.004	100	80	32	0.2	80	20	320	352
20	30	0.004	400	70	112	0.2	70	30	420	532
30	40	0.004	900	60	216	0.2	60	40	480	696
40	50	0.004	1600	50	320	0.2	50	50	500	820
50	60	0.004	2500	40	400	0.2	40	60	480	88o
60	70	0.004	3600	30	432	0.2	30	70	420	852
70	80	0.004	4900	20	392	0.2	20	80	320	712
80	90	0.004	6400	10	256	0.2	10	90	180	436
90	100	0.004	8100	0	0	0.2	0	100	0	0

Second computation, using (85),—

7.	ev/2 <sup>2</sup>	12.	n-r-1.	First Term.	wi <sup>2</sup> 2# <sup>2</sup>	P + Z.	#-F-1.	Second Term.	M <sub>r.+1</sub> .
0	4	0	9	0	20	I	9	180	180
I	4	1	8	32	20	2	8	320	352
2	4	4	7	112	20	3	7	420	532
3	4	9	6	216	20	4	6	480	696
4	4	16	5	320	20	5	5	500	820
5 6	4	25	4	400	20	6	4	480	88o
6	4	36	3	432	20	7	3	420	852
7 8	4	49	2	392	20	8	2	320	712
8	4	64	1	256	20	9	1	180	436
9	4	81	0	0	20	10	0	0	0

This second computation will, in general, be found more convenient than the first, since n and r are usually integers, and not very large.

31. When a uniform continuous load is coming upon one end of a girder, we may find the position of the foremost end of the load at the instant the moment at that end reaches its maximum value, by differentiating (81) with respect to b, and putting  $\frac{dM}{db} = 0$ . Thus,

$$\frac{dM}{db} = \frac{w'}{2l}(2lb - 3b^2) = 0,$$

$$\therefore b = \frac{2}{3}l. \tag{86}$$

32. The position of the foremost end of the live load when the moment is a maximum there for combined dead and live loads, is determined by differentiating (83), and making  $\frac{dM}{dh} = 0$ .

Thus, 
$$\frac{dM}{db} = \frac{w}{2l}(2elb - 3eb^2 + l^2 - 2lb) = 0,$$

$$\therefore b = \frac{l}{3e} \left\{ e - 1 \pm (e^2 + e + 1)^{\frac{1}{2}} \right\}, \quad (87)$$

where  $e = \frac{w'}{w}$ , and b = length of live load on one end of the girder.

Equation (87) is illustrated by the example in article 29, where  $e = \frac{0.8}{0.4} = 2$ ; and (87) gives b = 60.76, while (83) gives the corresponding moment 1056.31, a maximum.

33. Equation (55) expresses the moment due any uniform partial or complete continuous load, wb, at any loaded point, x. By differentiating (55), and putting  $\frac{dM}{dx} = 0$ , we may find the value of x, which gives the maximum moment. Thus,

$$\frac{dM}{dx} = \frac{w'b}{l}(l - a - \frac{1}{2}b) - \frac{1}{2}w'(2x - 2a) = 0,$$

$$\therefore x = a + b - \frac{b}{l}(a + \frac{1}{2}b), \tag{88}$$

which is the point of greatest moment due w/b.

34. Also, from (49), we may make

and find

$$\frac{dM}{dx} = \frac{1}{2}w(l - 2x) = 0,$$

$$x = \frac{1}{2}l,$$
(89)

which is the point of greatest moment due full continuous uniform load, as in equation (52).

35. If we add  $\frac{dM}{dx}$  from (55) to  $\frac{dM}{dx}$  from (49), and equate the sum to zero, we shall find

$$x = \frac{1}{w + w'} \left\{ (a + b)w' - (a + \frac{1}{2}b) \frac{w'b}{l} + \frac{1}{2}wl \right\}, \quad (90)$$

which is the point of greatest moment due both loads, w'b' and wl.

36. We will now consider the case of a girder having n panels, each  $= c = \frac{1}{n}$ ; and we will suppose the live load to consist of weights not all equal, nor spaced so as to conform to the panel points. Such a case is presented by a locomotive and train of cars.

Making use of equations (40) and (43), let us arrange formulæ convenient for this case.

If in these equations we put nc for l, and for a' and x write the proper multiple of c, we have the simultaneous moments due each weight, W, in its position at a panel point, as indicated in the following tabular arrangement:—

SMULTANEOUS MOMENTS.

W#-1	1.6W1	8 1 2 8	3c176.	1.M.7.	Sc 17.	* [o	# - 1 cW #-1
	•	:	•	:	•	:	• •
J.V.	R-6.W1	$\frac{9(\kappa-6)}{\kappa}_{cW_{\frac{1}{2}}}$	$\frac{3(n-6)}{4}$	4(x - 6) cW4	$\frac{5(m-6)}{2}cW_{\parallel}$	M 2 (9 - w)9	6 W =-1
*	#-5cW1	$\frac{2(n-5)}{n}_{CH}V_{3}$	$\frac{3(N-5)}{N}cH_{2}$	4(N- S)CH	5(8- 5) cW.	$\frac{5(n-6)}{n}cW_0$	SeW H-1
W,	$\frac{\pi-4cW_1}{4}$	$\frac{9(n-4)}{n}cW_{2}$	$\frac{3(n-4)}{n}cW_3$	4(n-4).W.	4(n-5),W	4(n - 6), W.	4cW n-1
¥,	# - 3cW1	$\frac{3(\kappa-3)}{\kappa}cW_3$	3(n-3)_cW <sub>3</sub>	3(*4)_cW_6	3(n-5)cWs	3(*-6)cW <sub>6</sub>	3cW =-1
W.	" = 2cW1	$\frac{\pi(n-2)}{n}_C \mathcal{W}_2$	$\frac{4(\pi-3)}{\pi}_{\mathcal{C}} \mu_{3}$	$\frac{n(\kappa-4)}{\kappa}_{CW_{\bullet}}$	$\frac{n(n-5)}{4}cW_b$	*(n-6)cW	2 W H - 1
₩1	" - 1cW	A CHE	# -3.W	*-4cW.	# - 5 W	# 6 W	EWR-t
Order.	M.W.	Mw.	M.W.	Mw.	Mw.	Mw.	MWn-:

Since the above moments are simultaneous, we may sum them, and thus find the moment at each joint due all the weights in their respective positions.

+(n-1)/ +(\*-3)17-4-/(a - a)+ W#-1 Equation (91). पास (a '2) IV. (m - 6) W +3 W n - b +2 W n - b + F. W. - 1 ž عائي 2 - 5 W. 2 - 3 W. 3 - 8 W. ٠ À +3W. SUM OF MOMENTS, ₩.\* +3/4/8-1 Z, + 37/4 + 4 27/4 + 4 21 ¥ = 1 • À ž + (2 - 12) % % 1 + (3 - 14) % % 1 + (3 - 14) % 1 + +(\* -6) W<sub>4</sub> +(\* -7) W<sub>7</sub> +(\* -8) W<sub>8</sub> ¥ G1# Order. - MK

Now, in this equation, (91), for the sum of the moments due all the weights, we may evidently put any weight in the place of any other, and suppose any number of the weights equal to zero.

Hence we may, by means of (91), find the momental effects of any load traversing the girder, at each of the equal intervals, c, in its progress.

Example. — Let the span = l = 100 feet; n = 10 = number of panels, each = c = 10 feet. Dead load = w = 0.5 ton per linear foot = cw = 5 tons at each panel point or apex, and  $2\frac{1}{2}$  tons on each abutment. Live load consists of two locomotives, each of the following lengths and weights:—

```
Between bearings of truck wheels, 5.75 feet.

Between bearings of second truck wheels and first driver, 8.50 feet = S_1.

Between bearings of drivers, 7.75 feet = S_2.

Between bearings of second drivers and first tender, 7.25 feet = S_2.

Between bearings of first and second tenders, 4.00 feet.

Between bearings of second and third tenders, 7.25 feet.

Between bearings of third and fourth tenders, 4.00 feet.

Total wheel base, 4.50 feet.
```

Between bearings of first tender and truck of second engine, 8 feet.

Total weight of tender,

Total weight of engine,

65,000 pounds = 21.00 tons.

Total weight on 2 pairs drivers, 42,000 pounds = 21.00 tons.

Total weight on 2 pairs truck,

23,000 pounds = 11.50 tons.

Weight on each pair truck wheels,

5.75 tons = k.

Weight on each pair drivers,

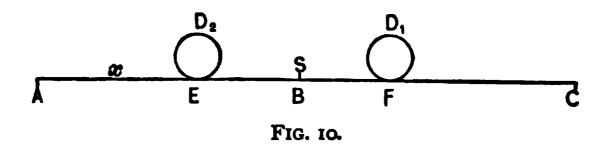
Weight on each pair tender wheels,

5.25 tons = t.

Suppose one girder carries these two locomotives.

We first find the greatest weight that can come upon a panel point or joint from the weights in adjacent panels.

Let  $D_1 = D_2$ , Fig. 10, be the equal weights on each pair of drivers, and take A, B, C, any three consecutive joints at the given interval, c feet; let x = AE, S = space between bases of drivers, it being less than c.



Then the weight at B, from drivers, is

$$\frac{x}{c}D_2 + \frac{2c - x - S}{c}D_1 = \frac{2c - S}{c}D, \qquad (92)$$

which is a constant, while the point B is anywhere in the space S.

If both drivers are between two consecutive joints, as AB, we have

$$\frac{x}{c}D_2 + \frac{x+S}{c}D_1 = \frac{2x+S}{c}D,$$

which is not a constant, but reaches its greatest value within the prescribed limits when x = c - S; that is, when

$$\frac{2x+S}{c}D=\frac{2(c-S)+S}{c}D=\frac{2c-S}{c}D.$$

Therefore  $\frac{2c-S}{c}D$  is the greatest pressure that can come upon any joint from the drivers of one engine.

Now, when the foremost driver,  $D_i$ , is at B, the second truck wheel is between B and C; and when the second driver is at

B, the first tender wheel is between A and B. In the former case the increment of weight at B from the truck would be  $\frac{c-S_i}{c}k$ ; in the latter case the increment at B from the tender would be  $\frac{c-S_2}{c}t$ . Therefore the first or second driver at B gives the greatest pressure at that point according as  $\frac{c-S_i}{c}k$ is greater or less than  $\frac{c-S_2}{c}t$ .

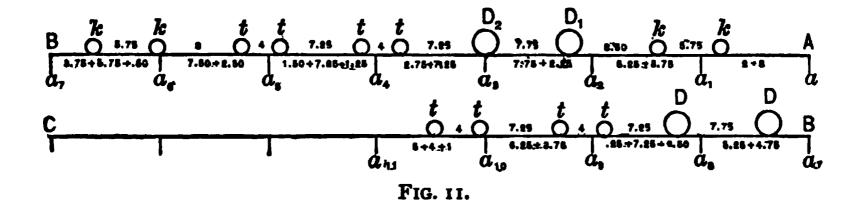
In the present example,

$$\frac{c - S_1}{c} k = \frac{10 - 8.5}{10} \times 5.75 = 0.86250,$$

$$\frac{c-S_2}{c}t = \frac{10-7.25}{10} \times 5.25 = 1.44375.$$

We will, therefore, find the pressures at points whose intervals are equal to c, when the second driver of the foremost engine is at one of these points. Let this point be the third, counting from the right-hand pier.

Then Fig. 11 shows the positions of all the wheels with reference to the joints at the equal intervals; and a simple calcu-



lation according to the principle involved in (38) and (39) gives the total pressure at each joint, which pressure is to be substituted for W in the equation (91).

In this position of the locomotives the pressures at the equal intervals due to the weights on the adjacent panels are—

```
1. At A, 0.200k
                          = 1.15000 tons.
 2. At a<sub>1</sub>, 1.425k
                          = 8.19375 tons.
 3. At a_2, 0.375k + 0.775D = 10.29375 tons.
 4. At a_3, 1.225D + 0.275t = 14.30625 tons.
 5. At a_4, 1.750t = 9.18750 tons.
 6. At a_5, 1.725t = 9.05625 tons.
 7. At a_6, 0.250t + 1.325k = 8.93125 tons.
 8. At a_7, 0.675k + 0.525D = 9.39375 tons.
 9. At a_8, 1.225D + 0.025t = 12.99375 tons.
10. At a_9, 0.250D + 1.600t = 11.02500 tons.
11. At a_{10}, 1.775t = 9.31875 tons.
12. At a<sub>11</sub>, 0.600t
                           = 3.15000 tons.
                       Total, 107.00000 tons.
```

Stopping with the second driver at  $a_3$ , we see that the hindmost tender truck has not yet come upon the girder. We compute, however, the moments due this load as it advances panel by panel, till the twelfth weight is upon the girder, and the first, second, and third have passed off; using indices  $M_1$ ,  $M_{1-2}$ , ...  $M_{3-11}$ ,  $M_{4-12}$ , to denote, inclusively, what panel weights produce the simultaneous moments opposite M.

			M <sub>(</sub>	MOMENTS DUE	LIVE LOAD.	Đ.			
JOINT.	150	<b>s</b> d.	3d.	4th.	5th.	6th.	7th.	8th.	9th.
$M_1$	10.350	9.200	8.050	6,900	5.750	4.600	3.450	2.300	1.150
$M_{1-8}$	82.944	83.950	73-457	62.962	52.469	41.975	31.481	20.988	10.494
$M_{1-3}$ .	166.244	229.550	210.918	180.788	150.656	120.526	90.396	60.264	30.131
$M_{1-4}$ .	275.362	407.662	436.023	384.449	320-374	256.299	192.214	128.150	64.075
$M_{1-5}$ .	324.107	556.339	645.508	631.741	536.031	428.826	321.619	214.413	107.207
$M_{1-6}$ .	362.482	634.402	814.447	851.429	785.470	637.576	478.182	318.789	159-394
$M_{1-7}$ .	390.676	692.040	902.242	1021.764	997.627	870.551	661.540	441.026	220.513
$M_{1-6}$	414.099	734.262	965.112	1115,400	1153.812	1059.161	861.575	582.051	291.026
$M_{1-9}$ .	460.532	791.226	1027.783	1175.130	1231.909	1196.815	1018.559	737.566	374-532
Ма-10 .	476.251	842.252	1078.316	1220.441	1273.263	1235.504	1105.881	833-193	457.565
M3-11 .	466.739	840.289	1103.589	1236.953	1276.377	1224.690	1086.042	863.712	478.322
M <sub>4</sub> -19 .	400.582	769.664	1045.558	1211.099	1246.805	1188.575	1041.132	803.026	473.043
M max	476.251	842.252	1103.589	1236.953	1276.377	1235.504	1105.881	863.712	478.322
Mw	225.000	400.000	525.000	600.000	625.000	600.000	525.000	400.000	225.000
M total .	701.251	i242.252	1628.589	1836.953	1901.377	1835-504	1630.881	1263.712	703.322

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	ΔM0-1	ΔM <sub>1</sub> -3	ΔΜ3-8	ΔM3-4	ΔM4-8	ΔM <sub>6</sub> – 6	ΔM <sub>6</sub> – τ	ΔM7-8	ΔW <sub>8</sub> - 9
-	10.350	-1.150	- 1.150	- 1.150	- 1.150	- 1.150	- 1.150	- 1.150	- 1.150
и	82.944	900.1	-10.494	<b>—10.494</b>	-10.494	10.494	- 10.494	10.494	10.494
က	166.244	63.306	-18.632	-30.131	-30.131	- 30.131	- 30.131	- 30.131	- 30.131
4	275.362	132.300	28.361	-51.574	-64.075	- 64.075	- 64.075	- 64.075	- 64.075
S	324.107	232.232	89.169	-13.767	<b>—95.710</b>	-107.207	702.201	-107.207	-107.207
9	362.482	271.920	180.045	36.982	-65.939	-147.894	-159.394	-159.394	-159.394
7	330.676	301.364	210.202	119.522	-24.137	-127.076	-209.011	-220.514	-220.513
<b>∞</b>	414.099	320.163	230.850	150.288	38.412	- 94.65I	-197.586	-279.524	-291.025
6	460.532	330.694	236.557	147.347	\$6.779	- 35.094	-178.256	-280.993	-363.034
01	476.251	366.001	236.064	142.125	52.822	- 37.759	-129.623	-272.688	-375.528
11	466.739	373-550	263.300	133.364	39.424	- 51.687	-138.648	-222.330	-385.390
13	400.582	369-082	275-894	165.541	35.706	- 58.230	-147.443	-238.106	-329.983
			_						

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INTERVAL.	0-1	1-8	<b>87</b>	7-8	2	9-9	6-7	7-8	8-9	9-10
Live load + .	476.251	373.550	275.894	165.541	\$6.779	ı	1	•	1	ı
Live load	ı	-1.150	-18.632	-51.574	-95.710	-147.894	-200.011	-280.993	-385.390	-478.322
Dead load.	225.000	175.000	125.000	75.000	25.000	- 25.000	- 75.000	-125.000	-175.000	-225,000
Maximum + .	701.251	548.550	400.894	240.541	81.779	ı	•	;	 :	
Maximum -	ſ	١	J	ı	-70.710	-172.894	-284.0II	-405.003	-t60.3m	
7.7	ı	•	1	1	-70.710	+ 70.710	1		260	-703.322
	703.322	560.390	405-993	284.011	172.894	-172.894	-284.011	-405.093	-360,300	1 06-1

The moments for dead load, given opposite Mw, have been computed by (65), whilst (91) has been used in finding the moments due the advancing load.

The differences are taken directly from the computed moments; and we must evidently use for each half-span the greatest difference due any interval, the load being supposed to travel either way.

37. Let us now suppose that this same live load of 107 tons is distributed uniformly over the 10 panels, so that W = panel weight = 10.7 tons. We then find by means of (65) the greatest moments due live load, and by means of (69) the greatest differences of moment due live load. Taking the moments due dead load as found above, we write:—

WEIGHT OF TWO LOCOMOTIVES UNIFORMLY DISTRIBUTED.

	DISTANCE FROM PIER.	10	20	80	40	50
	<b>r</b> .	•	I	2	3	4
2	$-\frac{\dot{W}c}{2n}(r+1)r \dots \dots$	•	-10.70	-32.10	-64.20	-107.00
2	Full live load, difference	481.50	374.50	267.50	160.50	53.50
3	Dead load, difference	225.00	175.00	125.00	75.00	25.00
4	Maximum difference +	706.50	549.50	392.50	235.50	<b>78.5</b> 0
5	Maximum difference —	-	-	-	-	82.00
6	Live load, M	481.50	856.∞	1123 50	1284.00	1337.50
7	Dead load, M	225.00	400,00	525.00	600.00	625.00
	Total M maximum	706.50	1256.00	1648.50	1884.00	1962.50
8	Total 1/2 maximum	700.50	1230.00	1040.30		-90-030
8	Distance From Pier.	60	70	80	90	100
8						
я	DISTANCE FROM PIER.	60	70	80	90	100
	Distance From Pier.	60 5 -160.50	<b>70</b> 6	80	90	100
I	DISTANCE FROM PIER. $r.$ $-\frac{Wc}{2\pi}(r+1)r \qquad$	60 5 -160.50	70 6 -224.70	80 7 -299.60	90 8 385.20	9 -481.50
1 2	DISTANCE FROM PIER. $r$ . $-\frac{Wc}{2\pi}(r+1)r$	60 5 -160.50 - 53.50	70 6 -224.70 -160.50	80 7 -299.60 -267.50	90 8 -385.20 -374.50	9 -481.50 -481 50
1 2 3	DISTANCE FROM PIER. $r$ . $-\frac{Wc}{2\pi}(r+1)r$ .  Full live load, difference.  Dead load, difference .  Maximum difference + .  Maximum difference —	60 5 -160.50 - 53.50	70 6 -224.70 -160.50	80 7 -299.60 -267.50	90 8 -385.20 -374.50	9 -481.50 -481 50
1 2 3 4	DISTANCE FROM PIER. $r$ . $-\frac{Wc}{2\pi}(r+1)r$ Full live load, difference  Dead load, difference  Maximum difference +	5 -160.50 - 53.50 - 25.00	70 6 224.70 160.50 75.00	80 7 -299.60 -267.50 -125.00	90 8 -385.20 -374.50 -175.00	9 -481.50 -481 50 -225.00
1 2 3 4 5	DISTANCE FROM PIER. $r$ . $-\frac{Wc}{2\pi}(r+1)r$ .  Full live load, difference.  Dead load, difference .  Maximum difference + .  Maximum difference —	5 -160.50 - 53.50 - 25.00 -185.50	70 6 -224.70 -160.50 - 75.00 -299.70	80 7 -299.60 -267.50 -125.00 - -424.60	90 8 -385.20 -374.50 -175.00 - -560.20	9 -481.50 -481 50 -225.00

A comparison of these maxima moments and differences with those just found for the natural distribution of the weights of these two locomotives, shows but one moment and one difference to be less for uniform load than for naturally distributed load. The extreme length of wheel base of these two united locomotives is  $2 \times 44.5 + 8 = 97$  feet.

It will be observed, that, when the second driver of the second engine is at a joint, the weight at that joint is greater than the weight we have used in the calculation of moments. But, when the second driver of the second engine is at a joint, the second driver of the first engine is 2.5 feet (see Fig. 11) from a joint; so that, assuming the coupled locomotives to travel either way, our calculation is correct.

We will close this section with an example including every kind of loading contemplated herein.

EXAMPLE. — Let 
$$W = 20.0 \text{ tons}$$
,  $a' = 50 \text{ ft.}$   
 $w = 0.4 \text{ tons}$ ,  $l = 100 \text{ ft.}$   
 $w' = 0.8 \text{ tons}$ ,  $b = 20 \text{ ft.}$ ,  $a = 40 \text{ ft.}$   
 $P = 10.0 \text{ tons}$ ,  $a' = 50 \text{ ft.}$ ,  $a = 30^{\circ}$ .

Find the moments and differences of moment for every ten feet throughout the girder.

In this calculation we use equations (40), (43), (49), (53), (55), (57), (47), and (48), with the following result:—

DISTANCE.	10	20	80	40	50	60	70	80	90	100
For W, M	100	200	300	400	500	400	300	200	100	0
For w, M	180	320	420	480	500	480	420	320	180	0
For w, M	80	160	240	320	360	320	240	160	80	0
For P, M	-25	<b>-50</b>	<b>-75</b>	-100	-125	-100	<b>-75</b>	50	-25	0
Total M	335	630	885	1100	1235	1100	885	630	335	0
Total dif	335	295	255	215	135	<b>—</b> 135	-215	<b>-255</b>	-295	-335

#### SECTION 3.

Horisontal Girder of One Span, with Fixed Ends. Effects of End
Moments.

38. If we suppose the simple girder, Fig. 9, not only supported at its ends, but also fixed by being built into the walls, or by means of forces applied to the sections AB and OE, to keep them from changing place as the beam inclines to yield to the other applied pressures, we then have a moment developed at each extremity of the girder, which will manifestly affect the normal moment due all other applied pressures at every cross-section.

Let us now find expressions for the momental effects at any point of the girder due to the given end moments, without attempting at present to formulate the value of these end moments, nor to determine whether they are simply sufficient to "fix" the ends of the girder.

Let AB, Fig. 12, represent a beam whose end moments are  $M_1$  and  $M_2$ .

#### F1G. 12.

Call the length of clear span I, and take VS any vertical section at the horizontal distance x from the left abutment. Let  $v_1$  = the vertical re-action, positive or negative, at B, due

to the moment  $M_1$  at A; and let  $v_2$  = the vertical re-action, positive or negative, at A, due to the moment  $M_2$  at B.

Then, taking moments about A and B, we have

$$M_1 = lv_1,$$
  $\therefore v_1 = M_1 + l,$   
 $M_2 = lv_2,$   $\therefore v_2 = M_2 + l.$ 

Call the moment at VS,

For 
$$v_1, -v_1(l-x) = -\frac{l-x}{l}M_1;$$
  
For  $v_2, v_2x = \frac{x}{l}M_2.$ 

Hence the moment at VS due to the two end moments acting in opposite directions is

$$M_x = \frac{l-x}{l}M_1 + \frac{x}{l}M_2 \stackrel{?}{=} \frac{M_2-M_1}{l}x + M_1,$$
 (93)

where both end moments tend to diminish the normal moment at the section VS, and are negative.

If, therefore, we apply the correction (93) to the moment found at any cross-section of a girder with free ends, which we have called the normal moment, we shall have the total moment, including the influence of the end or pier moments.

39. When  $c = l \div n =$  one of the equal panel lengths of the girder whose end moments are  $M_1$  and  $M_2$ , we may find the momental difference for any interval, c, due to  $M_1$  and  $M_2$ , by putting x + c for x in equation (93), and subtracting. Thus,

$$M_{x+c} = \frac{M_2 - M_1}{l}(x+c) + M_1,$$

$$\therefore \Delta M_c = M_{x+c} - M_x = (M_2 - M_1)\frac{c}{l}. \tag{94}$$

By means of (94) we may correct the normal difference of moments for the influence of the given pier moments.

The pier moments  $M_1$  and  $M_2$  are here supposed to be constant. The cases of their variation will be considered hereafter, when we come to formulate their values.

Example 1.—Let us suppose that the girder for which we have computed the maxima moments and differences of moment, in article 25, example 1, had, in addition to the pressures there given, been subjected to these end moments; viz.,

$$M_1 = -400$$
 foot-tons,  
 $M_2 = -500$  foot-tons.

From (93) we find decrements of moment:

$$\frac{400 - 500}{100}x - 400 = -410 \text{ when } x = 10$$

$$= -420 \text{ when } x = 20$$

$$= -430 \text{ when } x = 30$$

$$= -440 \text{ when } x = 40$$

$$= -450 \text{ when } x = 50$$

$$= -460 \text{ when } x = 60$$

$$= -470 \text{ when } x = 70$$

$$= -480 \text{ when } x = 80$$

$$= -490 \text{ when } x = 90$$

$$= -500 \text{ when } x = 100$$

From (94), or from the decrements just found, we have the constant decrement of difference,

$$\frac{400-500}{100} \times 10 = -10.$$

Applying these corrections to the tabulated maxima moments and differences in article 25, example 1, there results:—

x.		0	1	0	20	80	4	0	5	0	6	0 7	0	80	94	100
М		-40	0 1	130	540	830	10	00	10	50	9	80 7	90	480	50	<del>-500</del>
Difference	е.	{	530	418	31	4 2	18	+1;	30 70	+! -I!	1	<b>—238</b>		4 -4	38	-550

Example 2.— Let us suppose that the girder of example in article 37 has its right end extended 20 feet beyond the point of support, and has a weight,  $W_1 = 10$  tons applied at that extremity. What is the pier moment developed by the 10 tons and by the girder's own uniform weight, w = 0.4 ton per linear foot? And what is the effect of this pier moment on the normal moments and differences already found for the given pressures?

From (22), moment due W is  $-Wl = -10 \times 20 = -200$ . From (25), moment due w is  $-\frac{1}{2}wl^2 = -\frac{0.4 \times 20^2}{2} = -80$ . Moment at right pier  $= M_2 = -280$ . Moment at left pier  $= M_1 = 0$ . Whence, by (93), we have corrections of moment,

$$\frac{0 - 280}{100}x + 0 = -28 \text{ when } x = 10$$

$$= -56 \text{ when } x = 20$$

$$= -84 \text{ when } x = 30$$

$$= -112 \text{ when } x = 40$$

$$= -140 \text{ when } x = 50$$

$$= -168 \text{ when } x = 60$$

$$= -196 \text{ when } x = 60$$

$$= -196 \text{ when } x = 70$$

$$= -224 \text{ when } x = 80$$

$$= -252 \text{ when } x = 90$$

$$= -280 \text{ when } x = 100$$

From (94), correction for differences,

$$(0-280) \times \frac{10}{100} = -28.$$

Applying these corrections to the computed normal moments and differences, we find:—

x.	0	10	20	80	40	50		30	70	80	90	100
м	0	307	574	801	988	1095	9	)32	689	406	83	280
Dif. + . Dif	. 3	2	67 2	27 1	87 1	07 -	163	<del>-24</del> 3	<b>_283</b>	<b>-323</b>		-363

### CHAPTER IV.

STRAINS IN FRAMED OR BUILT GIRDERS, DEDUCED FROM THE MOMENTS OF THE EXTERNAL FORCES AND FROM THE SHEAR-ING-FORCES, AND FROM THESE COMBINED.

40. By the definition of statical moment, as given in article 9, it is the product of two numbers, — one representing the length of a straight line, the other the amount of force conceived to be applied at either end of the given straight line or lever arm, and to act in a line at right angles to that arm. If, therefore, H is the force or strain, and h the lever arm, the moment is

whose line of action is perpendicular to the arm h.

It hence appears that strains are deducible from moments; and, in order to apply this method to the determination of strains in girders of the most general description, let Fig. 13 represent one end of a framed girder, consisting of triangular panels, as ABD, CDF, EFH, etc., or of quadrilateral panels, ABDC, CDFE, EFHG, etc., whichever we choose to conceive them. Let the horizontal projection of each panel length, AC, CE, EG, etc., BD, DF, FH, etc., of both top and bottom chords, be equal to 2c; and let the apices, A, C, E, G, etc., be horizontally projected at the centres of the horizontal projections of

BD, DF, FH, etc., respectively. Let the inclinations of the segments of the top chord, AC, CE, EG, etc., to the horizon be  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$ , etc.; those of the segments of the bottom chord, BD, DF, FH, etc., to the horizon be  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$ , etc.; the inclination to the horizon of a Y web member, as BA, DC, FE, etc., be  $\phi_1$ ,  $\phi_2$ ,  $\phi_3$ , etc.; and the inclination to the horizon of a Z web member, as AD, CF, EH, etc., be  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$ , etc.: each angle of inclination of chord,  $\alpha$ ,  $\beta$ , to be measured from the horizontal drawn through the left end of the chord segment, and each angle,  $\phi$ ,  $\theta$ , to be measured from the horizontal through the lower extremity of the web member, as in trigonometrical notation.

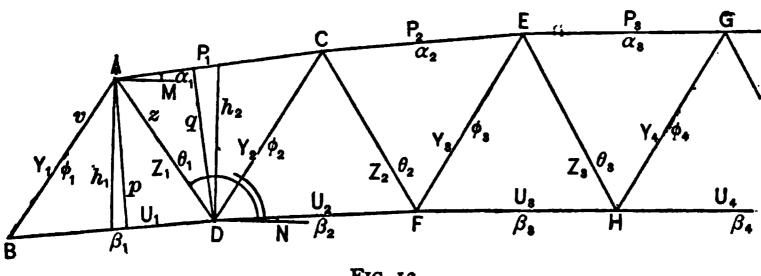


FIG. 13.

Assume, further, that each member of the structure is capable of resisting the strain that may come upon it, either of tension or compression; and, for distinction, call strains in compression positive, and tensile strains negative.

Also, the simultaneous forces acting at each apex are supposed to be in equilibrium, and the structure at rest. All the dimensions of the skeleton girder, as Fig. 13, are given, and may be varied so as to represent all the usual forms of girder, as illustrated below.

Let P symbolize the strain along any segment of the top chord; U the strain along any segment of the bottom chord; Y the strain along a Y web member whose slope is  $\phi$ , as defined

above, and length v; and Z the strain along a Z web member whose slope is  $\theta$ , and length s. Therefore  $p = -v \sin(\phi - \beta)$ , and  $q = z \sin(180 - \theta + \alpha) = z \sin(\theta - \alpha)$ , where p is negative, and represents the line drawn from any upper apex perpendicular to the chord opposite it, and q is positive, and denotes the length of the perpendicular drawn from any lower apex to the chord opposite.

Let  $H_r$  and  $H_{r+1}$  denote the two simultaneous horizontal strains at consecutive apices, whose difference,  $\Delta H$ , is the greatest of all differences of simultaneous horizontal strains for that interval; and let  $M_r$ ,  $M_{r+1}$ , be the corresponding moments, and  $h_r$ ,  $h_{r+1}$ , the heights or vertical distances from those apices to the axis of each chord opposite. Then  $H = M \div h$ , and  $\Delta H$  is the horizontal component of the strain developed in the diagonal or web member for the interval to which  $\Delta H$  belongs.

41. Suppose that the greatest moments due the given loading have been computed for the vertical section at right angles to the plane of the girder through each point, B, A, D, C, P, etc., Fig. 13; that is, at intervals each equal to c: and call these moments  $M_1$ ,  $M_2$ ,  $M_3$ , etc., at these consecutive intervals. Also, if the simultaneous moments which yield the greatest difference of horizontal strains at any two consecutive apices are different from these greatest moments, as may be the case for rolling loads, suppose such moments known.

We then have, from the figure and from the principles of articles 10 and 3,—

Strain along 
$$BD$$
,  $U_1 = M_2 \div p_1$ ; along  $AC$ ,  $P_1 = M_3 \div q_1$ .

 $DF$ ,  $U_2 = M_4 \div p_2$ ;  $CE$ ,  $P_2 = M_5 \div q_2$ .

 $FH$ ,  $U_3 = M_6 \div p_3$ ;  $EG$ ,  $P_3 = M_7 \div q_3$ .

Generally  $U = M_r \div p$ ;  $P = M_{r+1} \div q$ .

Strain along 
$$AB$$
,  $Y_1 = \Delta_1 H \div \cos \phi_1$ ; along  $AD$ ,  $Z_1 = \Delta_2 H \div \cos \theta_1$ .

 $CD$ ,  $Y_2 = \Delta_3 H \div \cos \phi_2$ ;  $CF$ ,  $Z_2 = \Delta_4 H \div \cos \theta_2$ .

Generally  $Y = \Delta_r H \div \cos \phi$ ;  $Z = \Delta_{r+1} H \div \cos \theta$ .

(97)

42. Shearing Forces and Strains. — If through any material body or structure we conceive a plane to pass, dividing the body into any two parts whatsoever, and assume either one of the two parts to be fixed in position, while the other part slides or tends to slide in any direction along this plane, then the force acting in a line parallel to the dividing-plane, and causing this sliding or tendency to slide, is called a shearing-force, and the strain on the particles of the body lying in this plane, resisting or tending to resist the shearing-force, is called the The amount of shearing-strain per unit of shearing-strain. the shearing-surface is its intensity; and the intensity at the instant of rupture (that is, at the beginning of actual sliding of one part of the body over the other along the shearing-plane) is the breaking shearing-strain, and is, in general, peculiar to each kind of material, and must be determined by experiment.

The published results of trustworthy experiments for determining the ultimate resistance of materials to shearing, are very meagre; and the following table, compiled from the works of two of the best authorities I know, viz., Professor Rankine and Mr. Bindon B. Stoney, is probably as worthy of confidence as any published records of the kind.

TABLE I.

Ultimate Resistance of Materials to Shearing, in Pounds, per Square Inch.

	<b>_</b>	
MATERIAL	Resistance to Shearing.	Remarks.
Metals.		
Cast-iron	27700 R.	Tensile strength ranges from 13400 to 29000. R.
Cast-iron	{ 17920 to } { 20160 S. }	"Substantially its tensile strength." S.
Wrought-iron .	55059 S.	Mean of 5 tests by Mr. Jones, punching-plates.
Wrought-iron.	50400 S.	Mean of 2 tests, Mr. Little, hammered scrap, inch punch.
Wrought-iron .	43456 S.	Mean of 4 tests, Mr. Little, hammered scrap, two-inch punch.
Wrought-iron .	50848 S.	Bar, 0.5 × 3 inches, punched both ways, Mr. Little, mean.
Wrought-iron .	48160 S.	2 bars, 1 × 3 inches, punched both ways, Mr. Little, mean.
Wrought-iron .	46144 S.	Flanged tire, 1.8 × 5 inches, edgewise, by Mr. Little.
Wrought-iron .	52192 S.	Rivet, 7 inch, Mr. Clark. Tensile strength 53760.
Wrought-iron .	45696 S.	Rivet, 7 inch, 2 plates, Mr. Clark.
Wrought-iron .	49952 S.	Rivet, 7 inch, 3 plates, Mr. Clark.
Wrought-iron .	50000 R.	•
Steel	63796 S.	Kirkaldy, rivet steel, tensile strength 86450.
Timber.		
Fir	592 S.	In direction of grain, Barlow.
Fir, red pine .	{ 500 to } 800 R.	
Fir, spruce	600 R.	
Fir, larch	{ 970 to } 1700 R.	
Oak	2300 R.	
Oak	4000 S.	Across grain, Rankine's deduction from Parsons's tests of English oak treenails.
Ash and elm .	1400 R.	

Abbreviations: R., Rankine; S., Stoney.

Instead of a plane cutting the body into two parts, we may conceive it cut into two separate parts by any cylindrical surface, and may suppose the sliding, or tendency to slide, to be in the direction of the generating line of the cylindrical surface, as in the case of a cylindrical punch.

43. From the definition of shearing-force, it follows, that if any girder, as Fig. 13, be cut by a vertical plane at right angles to its own plane, then the shearing force or strain at this vertical section is equal to the algebraic sum of the vertical components of all the forces impressed upon either side of this vertical plane. And these two algebraic sums of the vertical components of the forces impressed upon the opposite sides of this vertical plane will have contrary signs, and be numerically equal, except when a vertical force or weight is applied in the vertical plane itself, in which case the shearing-strains on opposite sides of the shearing-plane will differ by the value of this weight applied in the vertical plane.

Since the resultant of parallel forces is simply their algebraic sum, if the external forces applied to a girder are all vertical (that is, made up of the applied weights and the consequent vertical resistances of the piers), the shearing-force on either side of a vertical shearing-plane is merely the difference between the sum of the weights and the re-action of the pier on that side.

If, therefore, S denotes the shearing-force on either side of the shearing-plane, W being positive and denoting any weight, and V being the vertical re-action and negative, on the side chosen, we then have

$$S = V + \sum_{0}^{x} W, \tag{98}$$

where  $\Sigma_o^x W$  is the sum of all the weights between the shearingplane and the point of support having the re-action V; that is, of all the weights on the length x. In case of the semi-beam for the free end, V = 0, and

Sum of weights on 
$$l - x$$
,  $S = \Sigma_o^{l-x} W$ ,  
Sum of weights on  $l$ ,  $S = \Sigma_o^{l} W$ ; (99)

x being measured from the fixed end.

When the girder is supported at both ends, the re-actions due to a single weight, W, applied at the distance d, Fig. 9, from the left support, are, by equations (38) and (39),

At left support, 
$$V_1 = -W \frac{l-a'}{l}$$
;

At right support,  $V_2 = -W \frac{a'}{l}$ ;

calling them negative.

And for any number of different weights applied at different points,

$$V_{i} = -\Sigma \left( W \frac{l - a'}{l} \right)$$

$$V_{2} = -\Sigma \left( W \frac{a'}{l} \right).$$
(100)

Therefore for this case the shearing-strain at a vertical section distant x from the left support is

$$S = -\Sigma \left( W \frac{l - a'}{l} \right) + \Sigma_o^x W,$$
or
$$S = -\Sigma \left( W \frac{a'}{l} \right) + \Sigma_o^{l - x} W,$$
(101)

If upon the girder supported at both ends there are n-1

or

equal weights, W, at equal intervals,  $c = \frac{l}{n}$ , and  $\frac{1}{2}W$  upon each end of the girder on a pier, we have

$$V_{1} = -\frac{n}{2}W = V_{2},$$

$$S = -\frac{n}{2}W + (r + \frac{1}{2})W = \frac{1}{2}W(2r - n + 1);$$
(102)

the shearing-plane being at the  $r^{th}$  point of division counted from the left.

For a uniform continuous load, *lw*, upon a girder supported at its extremities,

$$V_{1} = -\frac{1}{2}hw = V_{2}.$$
At any point,  $x$ ,
$$S = -\frac{1}{2}hw + wx.$$
(103)

For a uniform continuous load, lw, upon a semi-girder at any point, x, measured from the fixed end, the shearing-strain is

and when 
$$x = 0$$
,
$$S = (l - x)w;$$

$$S = kw.$$
(104)

For any partial uniform continuous load, bw', Fig. 9, on a beam simply supported at its two ends, the re-actions of the piers are

$$V_{1} = -\frac{bw'}{l}(l - a - \frac{1}{2}b),$$

$$V_{2} = -\frac{bw'}{l}(a + \frac{1}{2}b);$$
(105)

which re-actions are identical with the shearing-strains for the unloaded parts of the beam.

But for the loaded part b, the shearing-strain is

$$S = -\frac{bw'}{l}(l - a - \frac{1}{2}b) + w'(x - a),$$
or
$$S = -\frac{bw'}{l}(a + \frac{1}{2}b) + w'(a + b - x).$$
(106)

If a = 0, and x = b, (106) becomes

$$S=\pm\frac{w'b^2}{2l},\qquad \qquad (107)$$

which is the shearing-strain at the foremost end of a uniform continuous load reaching to the left end of the beam supported at both ends. And equation (107) gives the greatest positive and the greatest negative value of S for this kind of load; since, in equations (106), x cannot be greater than b, and in the first of those equations S is an increasing function of x, while in the second S is a decreasing function of x.

The shearing-strain at any point, x, of the partial uniform continuous load on a semi-beam, Fig. 8, is

$$.S = w'(a + b - x);$$
 (108)

x being measured from the fixed end, and not being greater than a + b, nor less than a.

In order to simplify the application of equations (101) for the important case of a partial or complete uniform discontinuous moving-load, L, to be applied at equal intervals,  $c = \frac{l}{n}$ , along the girder, we proceed as in article 20, where we found the moments due such a load on a beam supported at both ends. Let  $r_1 - r_2 =$  the number of weights, L, on the beam

at any instant. Take r not less than  $r_2$  nor greater than  $r_1$ . Then, in the first of equations (101), we have

$$\Sigma a'^{\circ} = r_1 - r_2$$

and

$$\Sigma a' = c[(r_2 + 1) + (r_2 + 2) + (r_2 + 3) + \dots r_1]$$

$$= \frac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1)$$

for the first term. But in the second term,  $\sum_{0}^{x} L$ , we must take L no times for the left unloaded end of the beam,  $r - r_2$  times for the loaded part, and  $r_1 - r_2$  times for the unloaded part on the right end. Therefore

$$S = -\frac{L}{l}[(r_1 - r_2)l - \frac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1)]$$
 (109)

for the shearing-strain left of the load.

$$S = -\frac{L}{l}[(r_1 - r_2)l - \frac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1) - (r - r_2)l], \quad (110)$$

which is the shearing-strain between the points rc and (r + 1)c.

$$S = \frac{Lc}{2l}(r_1 - r_2)(r_1 + r_2 + 1) = \frac{L}{2n}r_1(r_1 + 1) \quad (111)$$

if  $r_2 = 0$ , and  $c = \frac{l}{n}$ ; and this is the shearing-strain at and

beyond the foremost end of a uniform discontinuous load reaching back to the left end of the beam.

44. The influence of end moments on the normal shearingstrains may be regarded as operating upon that term only of the shearing-strain which expresses the re-action of the pier.

Now, the pier moment  $M_2$ , acting at the right-hand pier, will affect the re-action  $V_1$  of the left pier by the amount  $-\frac{M_2}{l}$ ; and the pier moment  $M_1$ , acting at the left pier, will affect the

reaction  $V_2$  of the right pier by the amount  $-\frac{M_1}{l}$ . But, by the principles of article 10, the force  $-\frac{M_1}{l}$ , acting at the right end of the lever arm, l, induces a re-action,  $+\frac{M_1}{l}$ , at the left end of that arm, that is, in this case, at the left pier, where, consequently,

$$\Delta S = \Delta V_1 = \frac{M_1 - M_2}{l}.$$
Similarly
$$\Delta S = \Delta V_2 = \frac{M_2 - M_1}{l},$$
(112)

which are the increments of the shearing-strains due to the end moments, and are to be added algebraically to the shearing-strains found for the given load on the same beam simply supported at its two ends.

The values of  $M_1$  and  $M_2$  are here arbitrary, but will be determined for particular cases in subsequent chapters of this work.

45. To find the Shearing-Strain at any Vertical Section of a Girder (Fig. 13) in Terms of the Vertical Components of the Forces which are impressed upon the Shearing-Plane through the Members of the Girder cut by that Plane. — Using the notation already given in article 40, Fig. 13, and equations (3), we have, as the vertical component resulting from all the pressures on the left of each odd vertical plane, or the planes through the lower apices, B, D, F, etc.,

Left of 
$$B$$
,  $S_0 = V_1$ .  
Left of  $D$ ,  $S_2 = -P_1 \sin \alpha_1 + Z_1 \sin \theta_1 - U_1 \sin \beta_1$ ; Left of  $F$ ,  $S_4 = -P_2 \sin \alpha_2 + Z_2 \sin \theta_2 - U_2 \sin \beta_2$ ;  
Left of  $(2r + 1)^{th}$  apex,  $S_{2r} = -P_r \sin \alpha_r + Z_r \sin \theta_r - U_r \sin \beta_r$ ; (113) counting  $r$  on  $P$ ,  $Z$ , and  $U$ .

And on the left of each even vertical plane, or those through the upper apices, A, C, E, etc.,

Left of 
$$A$$
,  $S_1 = -P_0 \sin \alpha_0 - Y_1 \sin \phi_1 - U_1 \sin \beta_1$ ;  
Left of  $C$ ,  $S_3 = -P_1 \sin \alpha_1 - Y_2 \sin \phi_2 - U_2 \sin \beta_2$ ;  
Left of  $E$ ,  $S_5 = -P_2 \sin \alpha_2 - Y_3 \sin \phi_3 - U_3 \sin \beta_3$ ;  
Left of  $(2r)^{th}$  apex,  

$$S_{2r-1} = -P_{r-1} \sin \alpha_{r-1} - Y_r \sin \phi_r - U_r \sin \beta_r$$
; (114)

counting r on Y and U.

These values of S may be used to verify solutions by equations (96) and (97), as will be illustrated in some of the examples below.

46. Strains in all Members of a Girder determined from the Given Shearing-Forces. — Equilibrium of the system requires that at each apex, Fig. 13, the sum of the horizontal forces, as well as the sum of the vertical forces, shall vanish. Therefore at any lower apex we have

$$U_{r-1}\cos\beta_{r-1} - Z_{r-1}\cos\theta_{r-1} - Y_r\cos\phi_r - U_r\cos\beta_r = 0, \quad (115)$$
 and

$$P_{r-1}\cos\alpha_{r-1} + Z_r\cos\theta_r + Y_r\cos\phi_r - P_r\cos\alpha_r = 0 \quad (116)$$

at any upper apex.

The four equations, (113), (114), (115), (116), enable us to determine the four quantities,  $U_r$ ,  $P_r$ ,  $Y_r$ ,  $Z_r$ , in terms of  $P_{r-1}$ , and the given shearing-strains,  $S_{2r-1}$  and  $S_{2r}$ , if we use the auxiliary equation,

$$-P_{r-1}\cos\alpha_{r-1}=U_{r-1}\cos\beta_{r-1}-Z_{r-1}\cos\theta_{r-1},\quad (117)$$

expressing the equality of the horizontal strains at any lower apex, and at the point directly above it in the top chord.

Therefore, after solving and reducing,

$$U_r = \frac{S_{2r-1}\cos\phi_r - P_{r-1}\sin(\phi_r - \alpha_{r-1})}{\sin(\phi_r - \beta_r)},$$
 (118)

$$U_{r} = \frac{S_{2r-1}\cos\phi_{r} - P_{r-1}\sin(\phi_{r} - \alpha_{r-1})}{\sin(\phi_{r} - \beta_{r})}, \quad (118)$$

$$P_{r} = \frac{S_{2r}\cos\theta_{r} - U_{r}\sin(\theta_{r} - \beta_{r})}{\sin(\theta_{r} - \alpha_{r})}, \quad (119)$$

$$Y_r = \frac{-P_{r-1}\sin\alpha_{r-1} - U_r\sin\beta_r - S_{2r-1}}{\sin\phi_r}, \quad (120)$$

$$Y_{r} = \frac{-P_{r-1}\sin\alpha_{r-1} - U_{r}\sin\beta_{r} - S_{2r-1}}{\sin\phi_{r}}, \quad (120)$$

$$Z_{r} = \frac{P_{r}\sin\alpha_{r} + U_{r}\sin\beta_{r} + S_{2r}}{\sin\theta_{r}}. \quad (121)$$

Now, if we begin at the left end of the girder, Fig. 13, to compute,  $U_r$  becomes  $U_1$ , and  $P_{r-1}$  is zero; therefore (118) gives  $U_r$ : and with this value of  $U_r = U_r$  we at once find  $P_r = P_r$ , by (119), and similarly follow  $Y_r$  and  $Z_r$  from (120) and (121). A repetition of this process, putting the value of  $P_r$  just found, in the place of  $P_{\tau-1}$ , may be continued through the girder.

47. We will now give examples illustrating the determination of strains in open girders, first by the method of moments, and second by the method of shearing-strains, and will verify the solutions by equations (113) and (114).

Example 1.—Let B, Fig. 13, represent the unsupported end of a semi-girder, whose fixed end coincides with the vertical plane passing through E, and at right angles to the plane of the girder. Let the horizontal distance between consecutive apices, B, A, D, C, etc., = c = 10 feet, and the elevation of the apices, in feet, above the point B be as shown in the first line of the solution below. These elevations, with the horizontal distance c = 10 feet, furnish all the angles and lines required. If any apex is below B, its elevation is negative; and all angles and trigonometric functions follow the ordinary trigonometrical laws. At each apex, top and bottom, of this semi-beam, let a weight, W = 1 ton, be applied. Required the strains due this load in every member of the girder. The moment M is given by (35), where n = 5, l = 50, W = 1, and r takes the values 4, 3, 2, 1, 0, in succession.

LOGARITHMIC SOLUTION FOR DIMENSIONS AND STRAINS. — SEMI-BEAM. Method of Moments by Equations (96) and (97).

APBX.	Α.	D.	ပ	F.	শ্ৰ	H.
Elevation above B	18	ı	20	2.5	21	2.5
$\beta$	ı	0.05	ı	0.075	1	0
tan a	ı	•	0.1	1	0.05	ı
log tan $\beta$	ı	8.6989700	ı	8.8750613	•	ŧ
log sin β	ı	<b>28</b>	1	8.8738446	•	•
β	1	2° 51′ 45″	1	40 17, 21"	•	0
log tan a	ı		6	1	8.6989700	1
log sin a	1	ı	8.9978997	ı	8.6984422	1
	ı	ı	50 42' 41"	ı	20 51' 45"	1
tan •	81	1	1.9	•	1.85	1
tan θ	ı	-1.7	1	-1.75	J	1
log tan ∳ · · · · · · · ·	0.2552725	1	0.2787536	1	0.2671717	1
log sin ∳ · · · · · ·	9-9415891	1	9.9469040	1	9.9443377	ı
log cos ♦ · · · · • sos gol	9.6863166	1	9.6681466	1	9.677 1 666	•
•••••••••••	60° 56′ 43″	ı	62° 14′ 30″	ı	610 36' 25"	1
log tan $\theta$	ı	0.2304489*	1	0.2430380m	1	ı
log sin $\theta$	ı	9-9354741	1	9.9386408	1	ı
log $\cos \theta$ $\theta$ so $\log \theta$	ı	50	1	9.6956018	t	1
180° – θ · · · · · · θ – °81	ı	59° 32′ 4″	ı	60° 15′ 18″	ı	ı
• - B · · · · · · · ·	58° 4′ 58″		57° 57′ 9″	•	610 36' 25"	ı
$180^{\circ} - \theta + a$	1	65° 14' 45"	. 1	63° 7′ 3″	, 1	ı
c = 10 log 10	H		H	H	,	1
log v v gol	1.3136834	ı	1.3318534	ı	1.3228334	1
v = c ÷ cos ♦ · · · · ·	ı	1	1	)	1	1
log sin $(\phi - \beta)$	9.9288119	ı	9-9281953	1	99443377	1
	1.2424953m	•	1.2600487#	1	1.2671711*	1
$p = -v \sin(\phi - \beta) \cdot \cdot \cdot \cdot$	ı	ı	ì	ı	1	l

- Concluded.
SEMI-BEAM
AND STRAINS. —
DIMENSIONS AND
SOLUTION FOR ]
LOGARITHMIC

APEX.	¥	Ģ	ర	β <b>ε</b> ί	ជ	H.
log *		1.2949748	1	1.3043982	1	1
* = C ÷ cos θ · · · · ·	ı	1	1	ł	1	1
$\log \sin (180^{\circ} - \theta + a)$	I	9.9581397	ı	9-9503336	ı	ı
log q gol	ı	1.2531145	ı	1.2547318	•	1
$q = s \sin(180^{\circ} - \theta + a)$ .	ı	1	1	1	ı	1
	17.5	81	18.25	81	18.5	ı
M = -5(5-r+1)(5-r).	01-	-30	9	~ 100	9.1-1	1
log M	*	1.4771213#	1.77815134	<b>8</b>	2.17609134	1
$\log U_r = \log (M \div \rho) \cdot \cdot \cdot$	9.7 57 5047	1	0.5181026	1	0.9089202	1
<i>Cr.</i>	0.5721	ı	3.2969	ı	8.1081	ı
$\log P_r = \log (M \div q) \cdot \cdot \cdot$	1	0.2240068n	1	0.7452682m	1	1
Pr	1	-1.6749	1	-5.5625	ł	t
log h	1.2430380		1.2612629	1.2552725	1.2671717	1
$\log (M \div h) = \log H \cdot \cdot \cdot$	9.7 569620m	0.2218488	0.5168884#	0.7447275m	<b>0.9089196</b>	1
H $H$	-0.57143	99999:1-	-3.28767	<b>—5.55555</b>	-8.10810	ł
$\Delta H \cdots \cdots \cdots$	-0.57143	-1.09523	-1.62101	-2.26788	-2.55255	1
OOOOOOOOOOOOOOOOOOOOOOOOOOOOOOOOOOOO	9.7 569620m	0.0395014	0.20978562	0.3556202#	0.4069743%	1
$\log (\Delta H \div \cos \phi) = \log Y_r$ .	0.0706454*	ł	0.5416390%	ı	0.7298077#	1
	-1.1766	ı	-34805	ı	-5.3679	1
$\log (\Delta H \div \cos \theta) = \log Z_r$ .	1	0.3344762	1	0.6600184	1	1
Z	1	2.1601	1	4.5711	ı	1

Equations (113), (114). — Proof. — Shearing-Strains. — Tons.

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	*ŝ	1 4
	S	ကျ
)	S,	1 00
	Sı	H 1
		$-P_{r-1} \sin \alpha_{r-1} - Y_r \sin \phi_r - U_r \sin \beta_r$ $-P_r \sin \alpha_r + Z_r \sin \theta_r - U_r \sin \beta_r$

# LOGARITHMIC SOLUTION BY EQUATIONS (118) TO (121). Method of Shearing-Strains.

<b>7.</b>	1	28	8
фr	60° 56′ 43″	62° 14′ 30″	61° 36′ 25″
$a_{r-1}$	0 30 43	5° 42′ 41″	2° 51′ 45″
$\phi_r = a_{r-1}$	60° 56′ 43″	56° 31′ 49″	58° 44′ 40″
$eta_r$	2° 51′ 45″	4° 17′ 21″	0
$\phi_r = \beta_r$	58° 4′ 58″	57° 57′ 9″	61° 36′ 25″
$180^{\circ} - \theta_r$	59° 32′ 4″	60° 15′ 18″	-
$180^{\circ} - (\theta_r - \beta_r)  .  .  .  .$	620 23' 49"	64° 32′ 39″	_
$180^{\circ} - (\theta_r - a_r)  .  .  .$	65° 14′ 45″	63° 7′ 3″	_
$S_{2f-1}$	3 -4 45	2	E
$\log S_{27-1} \dots \dots$	0.	0.4771213	o.6989700
$\log \cos \phi_r$	9. <b>6</b> 863166	9.6681466	9.6771666
$\log S_{2r-1}\cos\phi_r  .  .  .  .$	9.6863166	0.1452679	0.3761366
$S_{2r-1}\cos\phi_r$	0.48564	1.3972	2.3776
$\log P_{r-1}$	-	0.2240067 <i>n</i>	0.7452619#
$\log \sin (\phi_r - a_{r-1}) \dots \dots$	-	9.9212585	9.9318958
$\log P_{r-1}\sin \left(\phi_r-a_{r-1}\right) \dots$	-	0.1452652n	0.6771577 <i>n</i>
$-P_{r-1}\sin\left(\phi_r-a_{r-1}\right)  .  .  .$	_	1.3972	4.7551
Numerator of (118)	0.48564	2.7944	7.1327
log num	9.6863166	0.4462948	0.8532540
$\log \sin (\phi_r - \beta_r)$	9.9288119	9.9281953	9.9443377
$\log U_r$	9-7575047	0.5180995	0.9089163
$U_r$	0.5721	3.2969	8.1081
$S_{2r}$	2	4	•••
$\log S_{2r}$	0.3010300	0.6020600	_
$\log \cos \theta_r$	9.7050252n	9 6956018 <i>n</i>	-
$\log S_{2r} \cos \theta_r$	0.0060552n	0.2976618n	-
$S_{2r}\cos\theta_{r}$	-1.01404	-1.9845	-
$\log \sin (\theta_r - \beta_r)$	9.9475214	9.9556478	-
$\log U_r \sin (\theta_r - \beta_r) \dots \dots$	9.7050261	0.4737473	_
$-U_r\sin(\theta_r-\beta_r)$	-0.50702	<b>—2.9768</b>	-
Numerator of (119)	-1.52106	<b>-4.9613</b>	-
log num	0.1821464n	0.6955955n	-
$\log \sin (\theta_r - a_r)$	9.9581397	9.9503336	-
$\log P_r$	0.2240067#	0.7452619#	_
$P_r$	-1.6749	<b>-5.5625</b>	_ •
$\log \sin a_{r-1}$	-	8.9978997	8.6984422

## LOGARITHMIC SOLUTION BY EQUATIONS (118) TO (121). — Concluded.

r.	1	28	8
$\log P_{r-1} \sin a_{r-1}.$ $P_{r-1} \sin a_{r-1}.$ $\log \sin \beta_r$ $\log U_r \sin \beta_r$ $U_r \sin \beta_r$ Numerator of (120) $\log \operatorname{num}.$ $\log \sin \phi_r$ $\log Y_r$ $Y_r$ From $P_{r-1} \sin a_{r-1} \operatorname{comes} P_r \sin a_r$ , Numerator of (121) $\log \operatorname{num}.$ $\log \sin \theta_r$	- 8.6984422 8.4559469 0.02857 -1.02857 0.0122339n 9.9415891 0.0706448n -1.1766 -0.16666 1.8619 0.2699587 9.9354741	9.2219064# -0.16666 8.8738446 9.3919441 0.24657 -3.0799 0.4885380# 9.9469040 0.5416340# -3.4805 -0.27777 3.9688 0.5986581 9.9386408	9.4437041#0.27777 0.9089163 04.7222 0.6741464# 9.9443377 0.7298087#5.3679
$\log Z_r$	0.3344846 2.1601	0.6600173 4.5711	-

N.B.—The method of shearing-strains, though applicable, is not conveniently used for live loads, since every change of load requires recomputation from the beginning.

- 48. Maxima Strains in the Web Members of an Open Girder, deduced from the Moments and Shearing-Forces combined, for Uniform Discontinuous Dead and Live Loads.
  - Let W = panel weight of dead load at (n 1) equidistant points,
    - L = panel weight of live load to be applied at the same points,
    - $M_W$  = moment at each panel point due dead load, by equation (65),
    - $M_L$  = moment due live load at its foremost end, by equation (64),

 $S_W$  = shearing-force at each panel point due dead load, by (102),

 $S_L$  = shearing-force at foremost end of live load due live load, by (111);

 $S_W + S_L =$ greatest shearing-force simultaneous with  $M_W + M_L$ ,

$$\frac{M_W + M_L}{h} = H = \text{simultaneous horizontal chord strain.}$$

Now, we have on the immediate right of any vertical plane: through an upper apex (Fig. 13),

$$U\cos\beta = -H_r = -P\cos\alpha + Z\cos\theta,$$
  
$$-P\sin\alpha + Z\sin\theta - U\sin\beta = S_r.$$

Whence, after eliminating U and P, there results,

$$Z = \frac{-H\frac{\sin(\beta - \alpha)}{\cos\beta} + S\cos\alpha}{\sin(\theta - \alpha)},$$
 (122)

where H and S belong to the vertical section through the upper extremity of the  $Z_{\theta}$  member, which joins the left end of the  $P_{\alpha}$  chord segment, to the right end of the  $U_{\beta}$  chord segment.

Similarly, on the immediate right of the vertical plane through the consecutive lower apex,

$$Y = \frac{H_{i} \frac{\sin(\beta_{i} - \alpha)}{\cos \alpha} - S_{i} \cos \beta_{i}}{\sin(\phi - \beta_{i})}, \qquad (123)$$

where  $H_1$  and  $S_2$  belong to the given vertical plane through the lower extremity of the Y member, which joins the left

end of the  $U_{i\beta_1}$  chord segment, to the right end of the  $P_a$  chord segment of equation (122).

It may here be observed, that according to our notation (article 40, Fig. 13), in any symmetrical girder,  $\theta$  in either half-span is the supplement of  $\phi$  in the corresponding panel of the other half-span; also  $\alpha$  and  $\beta$  in either half-span are respectively equal to  $-\alpha$  and  $-\beta$  of the corresponding panel of the other half-span. So s of the first half-span equals the corresponding v of the second.

•Example. — Uniform discontinuous dead and live loads. Let all that part of Fig. 13 which is on the left of the vertical line through E represent one of the equal half-spans of a girder supported at its two ends, B, and L not shown in the figure. Take the dimensions for each half-span the same as those already given in the example of article 47.

Let the dead load,  $W_1 = 4$  tons, be applied at each apex, top and bottom; and the live load,  $L_1 = 8$  tons, at the same points progressively. Each extreme apex may be supposed to bear  $\frac{1}{2}(W + L)$  when fully loaded; but this will only affect the resistances  $V_1$  and  $V_2$ , so far as the present method of computing strains is concerned. We may find greatest strains as follows:—

STRAINS DEDUCED FROM MOMENTS AND SHEARING-FORCES. — TONS.

	•	D.	ರ	Ħ.	ដ	H.		J.	ï
By (65), M W+L	540	ogó	1300	140	1500	1440	1200	og6	240
log M m+L · · · ·	2.7323938	2.9822712	3.0791812	3.1583625	3.1760913	ı	ı	1	1
log 1	1.2424953m	1	1.2600487n	ı	1.26717118	1	ı	ı	•
log q	,	1.2531145	1	1.2547318		ı	•	ı	•
$\log (M_{W+L} + p) \cdots$	I.4898985m	ı	1.8191325m	٠,	1.9089202n	ı	ı	ı	•
•	-30.896	1	-65.938	ı	-81.081	1	-65.938	ı	-30.896
$\log (M_{W+L} + q) \cdot \cdot \cdot$	•	1.7391567	ı	1.9036307	ı	1	1	ı	. 1
Maximum P	•	53.599	ı	80.097	1	80.097	•	\$3.599	1
By (65), M	81	330	430	480	200	8	420	330	180
By $(64)$ , $M_L \cdots$	2	192	336	480	8	673	673	576	360
$M_{W} + M_{L} \cdot \cdot \cdot \cdot$	253	512	756	8	0011	1159	1092	8%	540
By (102), S	-14	10	9	<b>?</b>	+2	9	+10	+1+	+18
By (iii), $S_L \dots \dots$	<b>6</b>	† 6	<b>↔</b>	<b>∞</b>	12	16.8	22.4	8.8	36
+ ST	-13.2	-7.6	-1.3	9+	+14	+22.8	+38.4	+42.8	<b>3</b> +
	17.5	82	18.25	<b>6</b> 0	18.5	81	18.25	891	17.5
$\log (M_W + M_L) \cdot \cdot \cdot$	3.4014005	2.7092700	2.8785218	2.9822712	3.0413927	3.0614525	3.0382226	2.9523080	2.7323938
log h	1.2430380	1.2552725	1.2612629	1.2552725	1.2671717	1.2552725	1.2612629	1.2552725	1.2430380
log H · · · · · ·	1.1583625	1.4539975	1.6172589	1.7269987	1.7742210	1.8061800	r.7769597	1.6970355	1.4893558
• • • • • • • • • • • • • • • • • • • •	2° 51′ 45″	ì	4 17 21"	1	0	ı	-4 17' 21"	ı	-2 51' 45"
•	1	S 42' 41"	1	2° 51′ 45″	1	-2° 5x' 45"	ı	-S 42' 41"	ı
B - B	-2 50′ 56″	1	1,92,36,	1	2 51' 45"	1	1 25, 20,	ı	-2° 51′ 45″
$\log \sin (\beta - a) \cdot \cdot \cdot \cdot$	8.6963739m	1	8.3961550	ı	8.6984422	•	8.3948002	ı	8.6984422m
log cos β, a. c	0.0005423	,	0.0012181	ı	d	1	0.0012181	ı	0.0005423
$\log [H \sin(\beta - \alpha) + \cos \beta]$ .	9.8552787m	1	0.0146320	1	0.4726632	1	0.1729780	1	o.1883403n
1st term of (122)	+0.7166		-1.0343	ı	-2.9694	1	-1.4893	ı	+1.5429
I of $I$	1.1205739n	o.8808136a	0.0791812m	0.7781513	1.1461280	I.3579348	1.5105450	1.6314438	1.7323938
log cos a	9.9978387	ı	9.9994578	1	9.9994578	1	9.9978387	1	ó
$\log (S_{W} + S_{L}) \cos \alpha \cdot \cdot$	1.1184126m	ı	0.0786390m	1	1.1455858	,	1.5083837	•	1.7323938

A. D.  -13.1345  -12.4179  1.09404828  -2.9281397  1.13590858  -13.674  -13.674  -13.674  -13.674  -13.674  -13.674  -14. es/ so/ 8.39480028  -15.674  -15.674  -16. es/ so/ 8.39480028  -17. es/ so/ -1							
	ပ	F.	Ä	H.		J.	I.
	-1.1985	ı	+13.9825	,	+32.2393	,	<b>7</b> 5+
	-2.2328	•	11.0131	1	30.7499	1	55.5429
	o.3488498m	1	1.0419095	1	1.4878438	1	1.7446285
<u> </u>	1	60° 15′ 18″	•	61° 36′ 25″	1	62 14' 30"	1
	63° 7′ 3′′	, 1	58° 44′ 40″	•	56° 31′ 49′′		60° 56′ 43″
	9.9503335	1	9-9318958	,	9.9212584	1	9.9415891
	o.3985163n	•	1.1100137	ı	1.5665854	•	1.8030394
8.3948002m - 8.3948002m - 0.0021613 - 9.8509590m - 0.7095	-2.503	1	+12.883	•	+36.863	1	+63.538
8.3948002m 0.0021613 9.8509590m -0.7095	ı	-2 51' 45"	ı	-1.25/30"	•	+3 50 56"	1
- 0.0021613 - 9.85095908 - 0.7095 - 9.9987819	ı	8.6984422m	•	8.396155om	•	8.6963739	1
9.8509590m 0.7095 	1	0.0005422	,	0.0005422	1	0.0021613	1
9.9987819	ı	O.4259831m	•	0.2028772n	•	0.3955707	1
9.9987819	ı	-2.6667	,	-1.5954	1	+2.4864	ı
TANDADA O	ı	ó	•	9.9987819	ı	9-9994577	,
一 またのかのかくのう	ı	0.7781513	,	1.3567167	•	1.6309015	1
- +7.5787	•	9	1	-22.7361	ı	-42.7466	ı
6.8692	,	-8.6667	ı	-24.3315	•	-40.2602	1
- 0.8369062	•	0.9378538m	1	1.3861689n	ſ	1.6048760m	1
•	62 14' 30"	1	61° 36′ 25′′	ı	60° 15′ 18′′	ı	59° 32′ 4′′
1	57 57 9"	ı	61° 36′ 25″	ŧ	64° 32′ 39″	1	62 23' 49"
- 9.9281953	ı	9.9443376	f	9.9556478	ı.	9-9475214	1
- 0.9087109	ı	0.99351628	ı	I.4305211R	t	1.6573546m	ı
+8.105	ı	-9.852	ı	-26.948	1	<b>-45.431</b>	1
63.538 -	36.863	ı	12.883	1	1	1	I
•	,	1	-9.852	•	-26.948	1	-45-431
1	1	•	12.883	1	36.863	1	63.538
45.43I	-26.948	1	-9.859	ſ	ŧ	1	1
_	-	1	-				

				,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,					
APEX.	Þ	D.	C	<u>'</u> ম্য	'n	Н.	<u>.</u>	J.	Ħ
By (65), M <sub>W</sub>	180	320	420	480	500	<b>4</b> 80	420	320	180
By $(64)$ , $M_L$	72	192	336	<b>\$</b> %	8	672	672	576	360
By (68), $M_{+1L}$	1	ţ	x68	288	<b>\$</b>	<b>\$</b> 80	<b>504</b>	***	288
$M_W + M_L \cdot \cdot \cdot \cdot \cdot$	252	512	756	960	1100	1152	1092	896	540
$M_{H}+M_{+1T}$	1	384	588	768	900	960	924	768	<b>\$</b> 68
$\log (M_{\overline{W}} + M_{L}) \cdot \cdot \cdot$	2.4014005	2.7092700	2.8785218	2.9822712	3.0413927	3.0614525	3.0382226	2.9523080	2.7323938
$\log (M_W + M_{+} \Pi) \cdot \cdot \cdot$	. 1	2.5843312	2.7693773	2.8853612	2.9542425	2.9822713	2.9656720	2.8853612	2.6702459
Jog &	1.2430380	1.2552725	1.2612629	1.2552725	1.2671717	1.2552725	1.2612629	1.2552725	1.2430380
$\log \frac{\eta}{M+MT} \cdots$	1.1583625	1.4539975	1.6172589	1.7269987	1.7742210	1.8061800	1.7769597	1.6970355	1.4893558
$\frac{M}{M+M+M}$ Sol	•	1.3290587	1.5081144	1.6300887	1.6870708	1.7269987	1.7044091	1.6300887	1.4272079
$\frac{M_{IP}+M_L}{k}=H_r$	14.400	28.4444	41.4247	53-3333	59.4595	64.	59.8356	49.7778	30.8572
$\frac{M_{R}+M_{+1L}}{k}=H_{+1}$	ı	21.3333	32.2192	42.6666	48.6486	53-3333	50.6301	42.6666	26.7429
$\Delta H = H_{+1} - H_r  .  .$	6.9333	3.7748	1.2419	-4.6847	-6.1262	-13.3699	-17.1690	-23.0349	-30.85 <b>73</b>
$\log \Delta H$	0.8409400	0-5768939	0.0940866	0.6706818#	0.7871912	1.1261282m	1.2347450m	1.3623864n	1.4893558
$\log \cos \theta$	9.7050252#	ı	9.6956018	1	9.67716667	ı	9.6681466#	ı	9.68631667
$\log (\Delta H + \cos \theta) \cdot \cdot \cdot$	1.1359148	ı	0.3984848	ı	1.1100246	1	1.5665984	ı	1.8030392
Z	-13.674	•	-2.503	ı	+12.883	١	+36.863	,	+63-538
log cosφ · · · · ·	ı	9.6681466	ı	9.6771666	•	9.6956018	ı	9.7050252	,
$\log (\Delta H + \cos \phi)$	,	0.9087473	1	0.9935152	•	1.4305264.1	1	1.6573612n	1
Y	ı	+8.105	ı	-9.852	1	-26.948	1	-45.431	ı

STRAINS FOUND FROM MOMENTS, FOR DEAD AND LIVE LOADS.

The chord strains, U and P, are to be found as before; their values being greatest when the two uniform loads cover the beam.

In the second line of this last solution,  $M_L$  is the moment due live load at its foremost end as that end passes the successive apices.

In the third line,  $M_{+iL}$  is the moment one interval beyond the foremost end, and simultaneous with  $M_L$ .

It is manifest, from what precedes, that we need compute the moments  $M_W$ ,  $M_L$ , and  $M_{+1L}$ , only when h is not constant; as, when h does not vary, we may find  $\Delta M_W$  and  $\Delta M_L$  by (71) and (69), whence  $\Delta H = \Delta M \div h$ .

49. We now proceed to classify girders according to the form which the general equations assume when particular values are assigned to one or more of their variables; first, recapitulating the general equations of the method of moments, and of the method of moments and shearing-forces.

From equations (95), (96), (97), (122), (123), we arrange GENERAL FORMULÆ.

$P = \frac{M_{r+1}}{q} = \frac{H_{r+1}}{\cos a}.$ $U = M_{(W+L)_r} \div p = H_{(W+L)_r} \div \cos \beta.$ $V = \frac{M_r}{p} = \frac{-H_r}{\cos \beta}.$ $Y = \frac{H_{r+1} \sin(\beta_1 - a) - S_{r+1} \cos a \cos \beta_1}{\cos a \sin(\phi - \beta_1)}.$		Метнор ог
$q = s \sin(\theta - a) = h_{r+1} \cos a.$ $H = \pm M \div h.$ $\Delta H = H_{r+1} - H_r, \text{ or }$ $\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$ $P = \frac{M_{r+1}}{q} = \frac{H_{r+1}}{\cos a}.$ $U = \frac{M_r}{p} = \frac{-H_r}{\cos \beta}.$ $Q = s \sin(\theta - a) = h_{r+1} \cos a.$ $H = \pm (M_W + M_L) \div h.$ $S = S_W + S_L.$ $P = M_{(W+L)(r+1)} \div q = H_{(W+L)(r+1)} \div \cos \theta.$ $U = M_{(W+L)_r} \div p = H_{(W+L)_r} \div \cos \beta.$ $V = \frac{H_{r+1} \sin(\beta_1 - a) - S_{r+1} \cos a \cos \beta_1}{\cos a \sin(\phi - \beta_1)}.$	Moments.	Moments and Shearing-Forces.
$Z = \Delta_{r+1} H \div \cos \theta. \qquad \qquad Z = \frac{\cos \beta \sin(\theta - a)}{\cos \beta \sin(\theta - a)}.$	$q = z \sin(\theta - a) = h_{r+1} \cos a.$ $H = \pm M \div h.$ $\Delta H = H_{r+1} - H_r, \text{ or }$ $\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$ $P = \frac{M_{r+1}}{q} = \frac{H_{r+1}}{\cos a}.$ $U = \frac{M_r}{p} = \frac{-H_r}{\cos \beta}.$ $Y = \Delta_r H \div \cos \phi.$	$q = s \sin(\theta - a) = h_{r+1} \cos a.$ $H = \pm (M_W + M_L) \div h.$ $S = S_W + S_L.$ $P = M_{(W+L)(r+1)} \div q = H_{(W+L)(r+1)} \div \cos a.$ $U = M_{(W+L)r} \div p = H_{(W+L)r} \div \cos \beta.$

- v = length of the Y web member making the angle  $\phi$  with the horizon, Fig. 13.
- z = length of the Z web member making the angle  $\theta$  with the horizon.
- p = length of perpendicular drawn from any upper vertex to the lower chord.
- = length of perpendicular drawn from any lower vertex to the upper chord.
- h = height of girder at any apex.
- $\alpha$  = inclination of any segment of the upper chord to the horizon, as angle CAM.
- $\beta$  = inclination of any segment of the lower chord to the horizon, as angle FDN.
- . $\phi$  = inclination to horizon of any Y web member, as angle CDN.
- $\theta$  = inclination to horizon of any Z web member, as angle ADN.
- W = panel weight of dead load.
- L = panel weight of live load.
- $M_{W}$  = moment due dead load.
- $M_L$  = moment due live load at its foremost end.
- $M_{W+L}$  = moment due dead load and full live load; that is, greatest moment for uniform loads.
- H = horizontal component of chord strain at a joint or apex.
- ΔH = difference of simultaneous horizontal components of chord strains at consecutive apices when this difference is greatest.
- $S_{W}$  = shearing-force due dead load on the immediate right of the shearing-plane.
- $S_L$  = shearing-force due live load at any point beyond its foremost end.
- P = strain in any segment of top chord.
- U = strain in any segment of bottom chord.
- Y = strain in any Y web member.
- Z = strain in any Z web member.

Count r always from the left, as indicated in the figures.

Now, although we have thus far considered each upper vertex to be horizontally projected midway between the horizontal projections of the lower vertices, this restriction is by no means necessary in the application of these equations, provided we compute the moments and the shearing-strains in accordance with the distribution of the loads, whatever that may be.

Length.	a.		y.		z. Z.		<i>v</i> .	
Strain.	Р.							
Class.	Top Chord.	a,	Tension Web Member.	φ.	Compression Web Member.	θ.	Bottom Chord.	β.
I	Inclined .	a	Inclined .	φ	Inclined.	θ	Inclined .	В
II	Inclined .	a	Inclined.	φ	Inclined.	θ	Horizontal,	0
III	Horizontal,	0	Inclined.	ø	Inclined.	θ	Inclined .	B
IV	Horizontal,	0	Inclined.	ø	Inclined.	θ	Horizontal,	0
v	Inclined .	Œ	Inclined.	φ	Vertical.	900	Inclined .	ß
VI	Inclined .	a	Vertical.	900	Inclined.	θ	Inclined .	β
VII	Inclined .	a	Inclined.	ø	Vertical.	900	Horizontal,	0
VIII	Horizontal,	0	Inclined.	ø	Vertical.	900	Inclined .	B
IX	Horizontal,	0	Inclined.	ø	Vertical.	900	Horizontal,	0
X	Inclined .	a	Vertical.	900	Inclined.	θ	Horizontal,	0
XI	Horizontal,	0	Vertical.	90°	Inclined.	θ	Inclined .	ß
XII	Horizontal,	0	Vertical.	900	Inclined .	θ	Horizontal,	0

THE TWELVE CLASSES OF GIRDERS OF SINGLE SYSTEM.

The conditions yielding the twelve classes may be briefly stated thus:—

With regard to 
$$\alpha$$
 and  $\beta$  we may have 
$$\begin{cases} \text{neither,} \\ \text{the one,} \\ \text{the other,} \\ \text{both,} \end{cases} = 0; \text{ 4 conditions.}$$
With regard to  $\theta$  and  $\phi$  we may have 
$$\begin{cases} \text{neither,} \\ \text{the one,} \\ \text{the other,} \end{cases} = 90^{\circ}; \text{ 3 conditions.}$$

Combining these conditions gives twelve classes and no more.

## CLASS I. - ALL MEMBERS BUT ONE INCLINED.

Use General Formulæ.

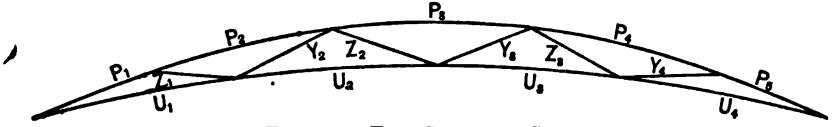


Fig. 14. — The Crescent Girder.

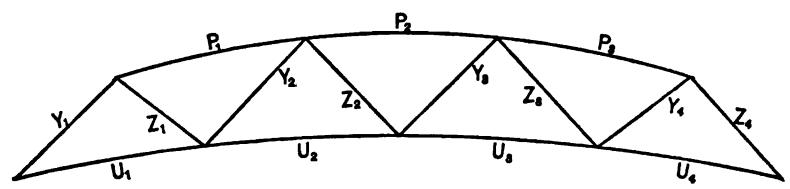


FIG. 15.—THE TRUNCATED CRESCENT.

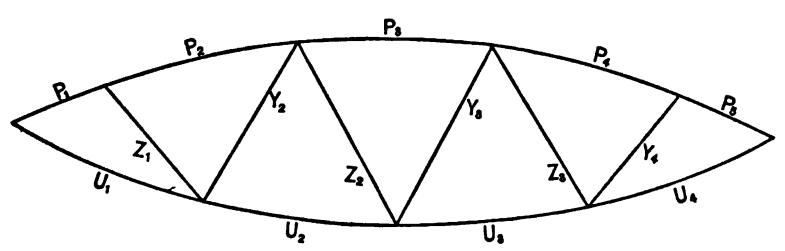


Fig. 16.—The Double Bow, or Brunel Girder.

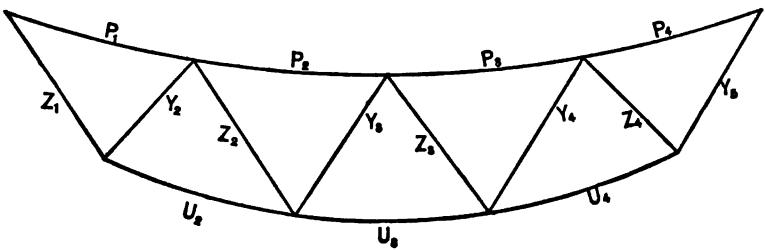


FIG. 17. - INVERTED TRUNCATED CRESCENT.

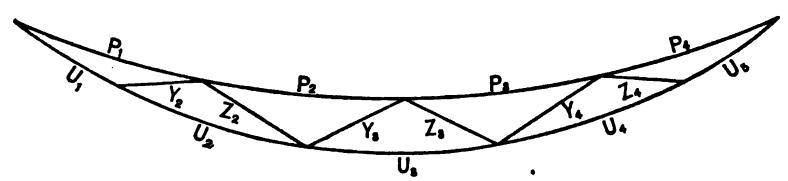


Fig. 18. - Inverted or Suspended Crescent.

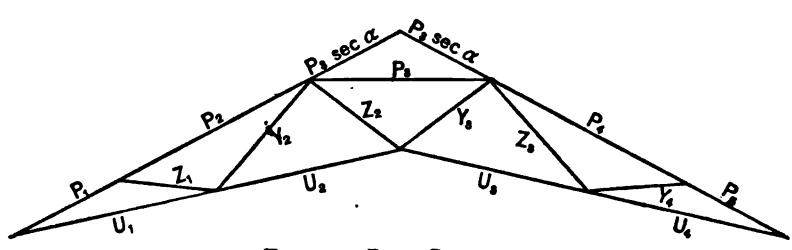


Fig. 19.—Roof Principal.

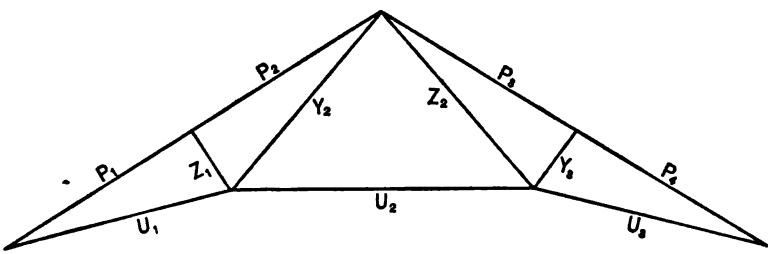


FIG. 20. — ROOF PRINCIPAL.

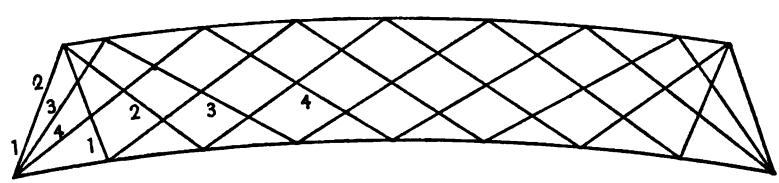


FIG. 21. - BENT GIRDER OF FOUR SYSTEMS.

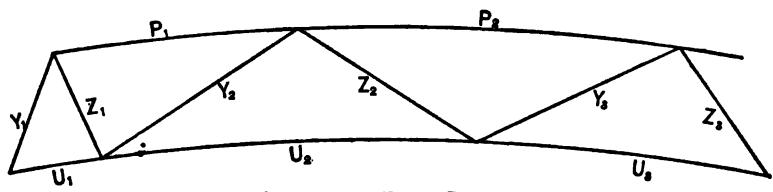


FIG. 21a. - FIRST SYSTEM.

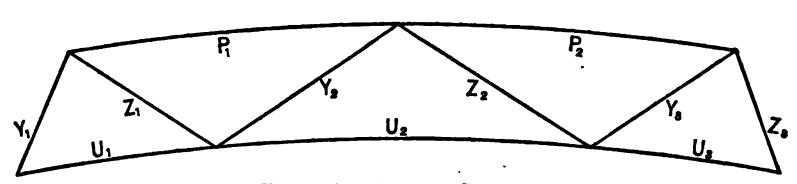


FIG. 216. — SECOND SYSTEM.

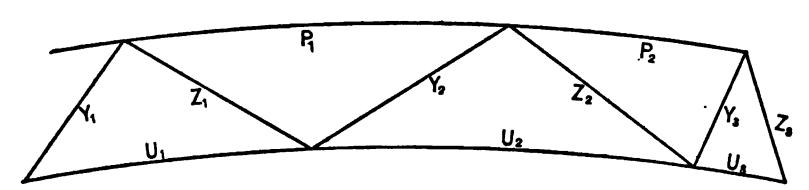


FIG. 21c. - THIRD SYSTEM.

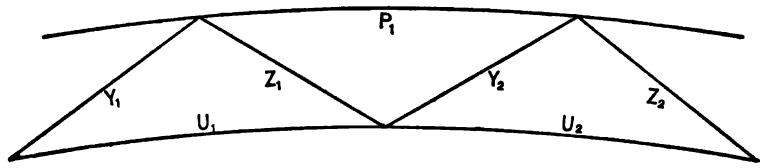


FIG. 21d. - FOURTH SYSTEM.

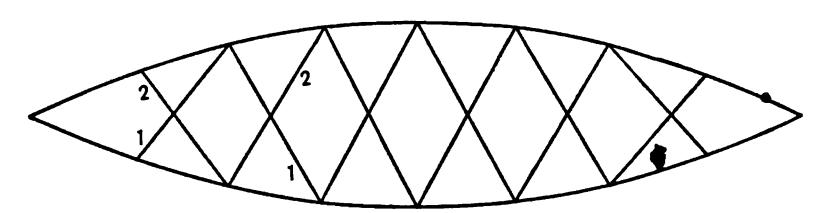


FIG. 22. — DOUBLE BOW OF TWO SYSTEMS.

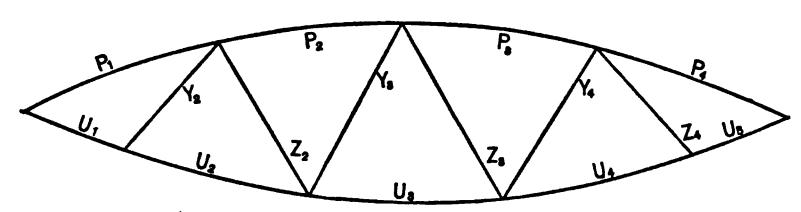


FIG. 22a. - FIRST SYSTEM. FIG. 16 SHOWS THE SECOND SYSTEM.

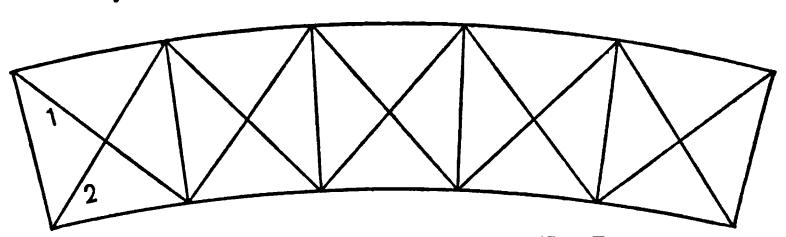


Fig. 23.—Segment of Roof Principal. (See Fig. 71.)

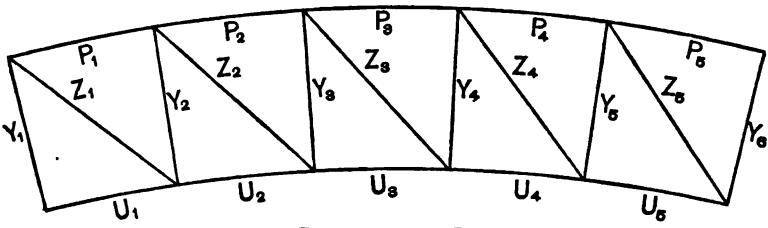


Fig. 23a. - Diagonals in Compression.

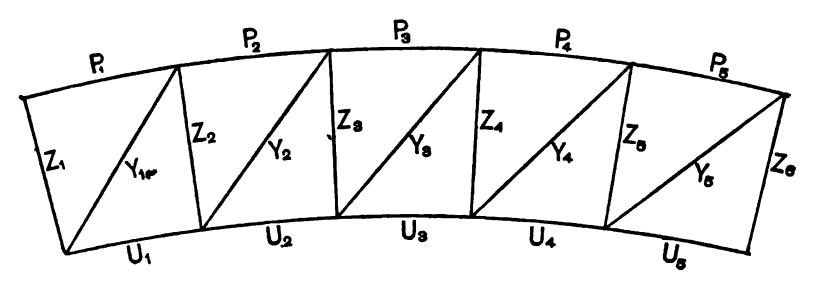


Fig. 236. - Diagonals in Tension.

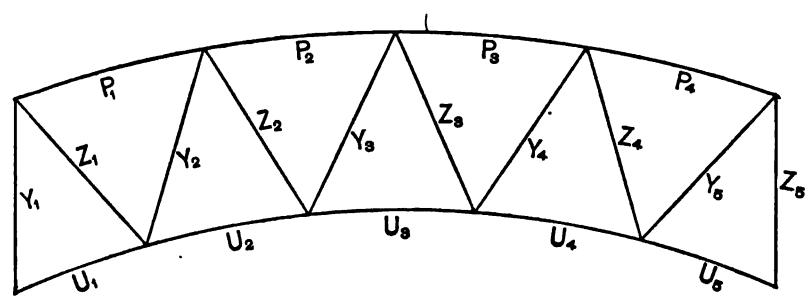


Fig. 24. — Parallel Chords. Triangular.

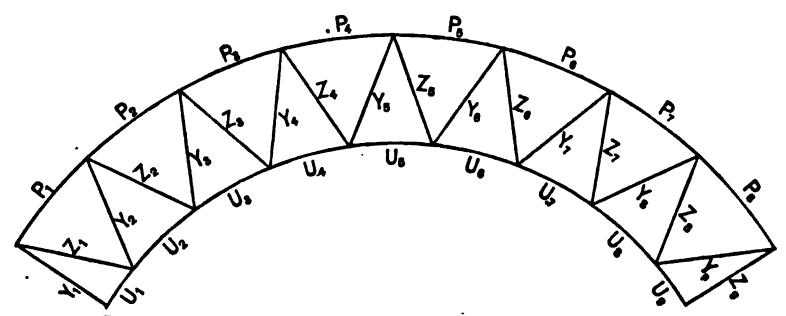
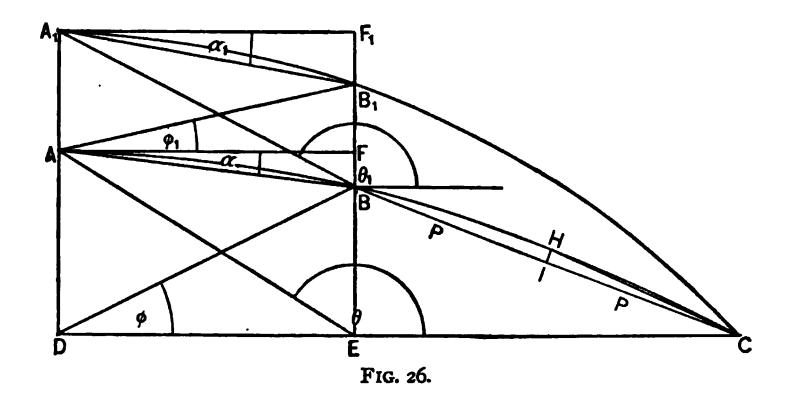


Fig. 25.—The Braced Arch. St. Louis Bridge System. (See Article.)

and

Although we have supposed the linear dimensions of the girder known, we will now give a mode of finding them from the known length l, and central height h, in case of girders having either chord, or both chords, circular or parabolic.



1st, Lower chord horizontal, and upper chord circular, as ABCED, Fig. 26. Let l = 2DC = length of bottom chord, h = AD = central height of girder. Then the radius

$$R = \frac{l^2 + 4h^2}{8h}.$$
 (124)

Take D, the centre of the chord of any arc, as ABC or  $A_1B_1C$ , for the origin of rectangular co-ordinates; the axis of x being horizontal, and that of y being vertical. Then the equation to the curve ABC is

$$(y + R - h)^2 = R^2 - x^2,$$
 
$$BE = y = h - R \pm \sqrt{(R + x)(R - x)}, \quad (125)$$

which is the height of the bowstring girder at any point, E, whose distance DE from the origin is x.

Thus, if l = 100, and k = 10, we have  $R = \frac{100^2 + 4 \times 10^2}{8 \times 10}$  = 130; and from (125) we find,—

When 
$$x = 0$$
,  $y = 10 = h$ .  
 $x = 10$ ,  $y = 9.6148$ .  
 $x = 20$ ,  $y = 8.4523$ .  
 $x = 30$ ,  $y = 6.4912$ .  
 $x = 40$ ,  $y = 3.6931$ .  
 $x = 45$ ,  $y = 1.9631$ .  
 $x = 50 = \frac{1}{2}$ ,  $y = 0$ .

The length of any diagonal, DB, whose inclination to the horizon  $= \phi$ , is

$$v = \frac{BE}{\sin \phi} = \frac{DE}{\cos \phi};$$

or, in general,

$$v = y_r + \sin \phi = \Delta x + \cos \phi, \qquad (126)$$

 $\Delta x$  being one panel length.

The length of any diagonal, AE, whose inclination to the horizon =  $\theta$ , not acute, is

$$z = \frac{AD}{\sin \theta} = -\frac{DE}{\cos \theta};$$
or, in general,
$$z = y_{r-1} + \sin \theta = -\frac{\Delta x}{\cos \theta}.$$
(127)

The length of any chord, AB, whose inclination to the horizon =  $\alpha$ , is

$$a = \frac{DE}{\cos \alpha} = \frac{FB}{\sin \alpha};$$
or, in general,
$$a = \frac{\Delta x}{\cos \alpha} = \frac{\Delta y}{\sin \alpha}.$$
(128)

From (126), 
$$\frac{\sin \phi}{\cos \phi} = y_r + \Delta x = \tan \phi. \tag{129}$$

From (127),

$$\frac{\sin\theta}{\cos\theta} = -y_{r-1} + \Delta x = \tan\theta. \tag{130}$$

From (128),

$$\frac{\sin\alpha}{\cos\alpha} = \Delta y + \Delta x = \tan\alpha. \tag{131}$$

Whence  $\phi$ ,  $\theta$ , and  $\alpha$ , and their sines and cosines, may be taken from a table of natural or logarithmic circular functions.

To determine the length of any part of the circular arc ABC, or of the whole arc, find the angle at the centre corresponding to the given chord of the required arc; then the required length of arc is to the whole circumference as the angle at the centre is to four right angles. Thus, if C denotes the angle at the centre whose chord is the span l = 100, we have

$$\sin\left(\frac{1}{2}C\right) = \frac{\frac{1}{2}l}{R} = \frac{50}{130} = \sin 22^{\circ} 37' 11''.5,$$

$$\therefore C = 45^{\circ} 14' 23'' = 45^{\circ}.239722.$$

Circumference = 360 degrees.

Length of circumference =  $2\pi R = 2 \times 3.14159 \times 130$  = 816.8134,

.. Required arc 
$$2ABC = \frac{45.239722}{360} \times 816.8134 = 102.645$$
.

Or, the length of the arc subtended by a given chord may be taken from a table constructed for the purpose.

2d, Both chords or flanges circular, as  $ABCB_1A_1$ , Fig. 26, where the chords meet at the ends of the girder; or, as Figs. 15 and 17, where the chords do not meet at the ends.

If the curves meet, as in Figs. 14, 16, 18, and 26, then l in equation (124) will be common to both arcs, ABC,  $A_1B_1C$ ; but the central heights of the two arcs will be h = AD, and  $h_1 = A_1D$ . Hence, for the upper curve, the radius

$$R_{\rm r}=\frac{l^2+4h_{\rm r}^2}{8h_{\rm r}}.$$

If, as in Fig. 15, the curves do not intersect at the ends, then, for each curve, l will be the chord subtended by the arc, h will be the central height of each arc above its chord, and the origin of co-ordinates for each curve will be at the centre of its own chord.

The ordinates y, corresponding to the same values of x, are to be found for each curve by (125); and if  $y_1 =$  an ordinate to the upper curve, and y = the corresponding ordinate to the lower curve, and e = the difference in height of the two origins, then  $y_1 + e - y =$  the height of girder at any point, x. And when e = 0, as in Fig. 26, the height of the girder  $ABCB_1A_1$  at any point, B, is  $BB_1 = y_1 - y$ .

$$FB = BE - AD = \Delta y, \text{ in general.}$$

$$F_{i}B_{i} = B_{i}E - A_{i}D = \Delta y_{i}, \text{ in general.}$$

$$DE = AF = A_{i}F_{i} = \Delta x, \text{ in general.}$$

$$\tan \alpha = \frac{\Delta y}{\Delta x}, \quad \tan \alpha_{i} = \frac{\Delta y_{i}}{\Delta x}.$$

$$AA_{i} + BF = y_{i,r} - y_{r+1},$$

$$B_{i}F = BB_{i} - FB = y_{i,r+1} - y_{r}.$$

$$\tan \theta = -\frac{y_{i,r} - y_{r+1}}{\Delta x}, \quad \tan \phi = \frac{y_{i,r+1} - y_{r}}{\Delta x}.$$

From these tangents,  $\alpha$ ,  $\alpha$ ,  $\theta$ ,  $\phi$ , and their sines and cosines, are to be found as before. We then have

$$v = AB_1 = \frac{y_{r_{r+1}} - y_r}{\sin \phi} = \frac{\Delta x}{\cos \phi}, \qquad (132)$$

$$z = A_{x}B = \frac{y_{x_{r}} - y_{r+x}}{\sin \theta} = -\frac{\Delta x}{\cos \theta}, \qquad (133)$$

$$a = AB = \frac{\Delta y}{\sin \alpha} = \frac{\Delta x}{\cos \alpha}, \quad (134)$$

$$a_1 = A_1 B_1 = \frac{\Delta y_1}{\sin \alpha_1} = \frac{\Delta x}{\cos \alpha_1}$$
 (135)

3d, When the curvature of one or both chords of the girder is parabolic, we proceed as in case of the circular chords just discussed, except in finding the ordinates and length of the curve, which only, therefore, we need now determine.

Let the curves, Fig. 26, now be parabolas, whose vertices are at A and A, respectively. Take the origin of rectangular co-ordinates, as before, at D; the axis of x being horizontal, and that of y vertical. Then the equation to the curve ABC is

$$y = \left(1 - \frac{4x^2}{l^2}\right)h; \qquad (136)$$

to the curve  $A_1B_1C_2$ 

$$y_i = \left(1 - \frac{4x_i^2}{l^2}\right)h_i.$$
 (137)

For the same value of x, (136) and (137) give

$$h_x = y_1 - y = \left(1 - \frac{4x^2}{l^2}\right)(h_1 - h),$$
 (138)

which is the height of the girder at any point whose distance is x from the centre or origin.

Thus, if l = 100, and h = 10, (136) gives, —

When 
$$x = 0$$
,  $y = 10 = h$ .  
 $x = 10$ ,  $y = 9.6$ .  
 $x = 20$ ,  $y = 8.4$ .  
 $x = 30$ ,  $y = 6.4$ .  
 $x = 40$ ,  $y = 3.6$ .  
 $x = 45$ ,  $y = 1.9$ .  
 $x = 50 = \frac{1}{4}$ ,  $y = 0$ .

And if  $h_1 = A_1D = 20$ , h = AD = 10, and l = 2DC = 100, equation (138) gives the heights of girder  $ACA_1$  as below:—

When 
$$x = 0$$
,  $h_0 = 10 = h_1 - h$ ;  
 $x = 10$ ,  $h_{10} = 9.6$ ;  
 $x = 20$ ,  $h_{20} = 8.4$ ;  
 $x = 30$ ,  $h_{30} = 6.4$ ;  
 $x = 40$ ,  $h_{40} = 3.6$ ;  
 $x = 45$ ,  $h_{45} = 1.9$ ;  
 $x = 50 = \frac{1}{2}l$ ,  $h_{50} = 0$ ;

which are the same as the heights of the girder ADC just found, since  $h_1 = 2h$ , and, from (136) and (137),

$$\frac{y}{y_{i}} = \frac{h}{h_{i}}; \qquad (139)$$

that is, the ordinates to the two curves, for the same value of x, are proportional to their central heights.

The length of the parabolic arc, in terms of the chord l and the central height h, is

$$S = \frac{1}{2}(l^2 + 16h^2)^{\frac{1}{2}} + 0.287823\frac{l^2}{h}\log\frac{4h + (l^2 + 16h^2)^{\frac{1}{2}}}{l}, \quad (140)$$

where log means the common logarithm.

If l = 100 feet = span, and h = 10 feet = central height, of parabolic arc, then (140) gives S = 102.606 feet.

From these examples it appears, that, when the curvature is small, there is but little difference between the ordinates and arc of the circular and the ordinates and arc of the parabolic girder of the same central height and span.

Instead of these exact determinations of the linear dimensions of a girder, the figure may be drawn to a scale, and the length of each member measured, where greater accuracy is not required.

It is proper to observe here, that, in all cases of curved flange, the line of action of the flange strain, P or U, is the chord of the arc between adjacent apices, and not the arc itself. When, therefore, either flange of a girder is curved, and not polygonal, there is developed midway between adjacent apices in the same flange a deflecting force tending to increase the curvature of a compressed flange, and to diminish the curvature of a flange in tension.

For the amount of this deflecting force F we have, if P is the strain along the chord BIC, Fig. 26,

$$F = 2P \tan HCI. \tag{141}$$

Or, if C is the angle at the centre of the circle whose chord is BC, then

$$\tan HCI = \frac{\operatorname{ver-sin} \frac{1}{2}C}{\sin \frac{1}{2}C} = \frac{1 - \cos \frac{1}{2}C}{\sin \frac{1}{2}C} = \operatorname{cosec} \frac{1}{2}C - \cot \frac{1}{2}C,$$

$$\therefore F = 2P(\operatorname{cosec} \frac{1}{2}C - \operatorname{cot} \frac{1}{2}C); \qquad (142)$$

and the strain along the chord HC of each half of the arc BC is

$$P' = P + \cos HCI. \tag{143}$$

Similarly, in cases like  $P_3$ , Fig. 19, there is a deflecting force generated at the ridge equal to

$$F = 2P_3 \tan \alpha;$$

and the strain along the upper segment of each rafter is, as indicated,  $P_3 \div \cos \alpha$ .

The bending-moment due to this deflecting force is given by equation (46),

$$M = \frac{1}{4}Fa, \qquad (144)$$

where a is the length of the chord BC.

The amount of material required to neutralize this moment will be determined in the sequel. But it is already manifest that the least amount of resisting material will be required when the line of pressure coincides with the axis of the resisting member.

Multiple or compound web systems, as those represented in Figs. 21 and 22, may be separated into the single systems of which they are composed, when the sum of all the strains found for the same member will be the strain sought for that member.

Class II. — Bottom Chord horizontal, other Members inclined.  $\beta = 0$ .

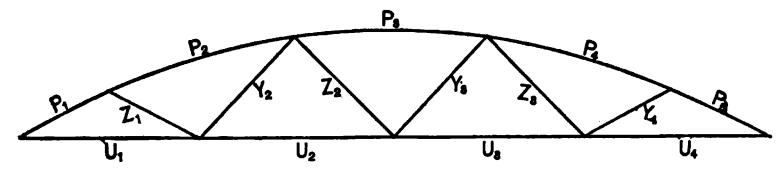


Fig. 27. — The Parabolic or Circular Bowstring.

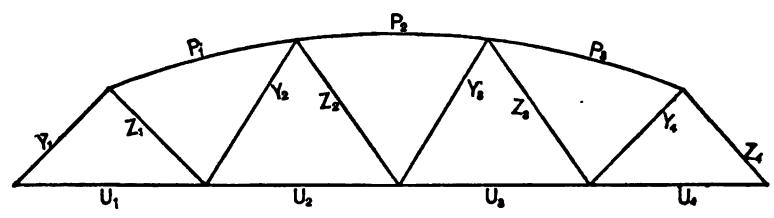


Fig. 28.—The Truncated Bowstring.

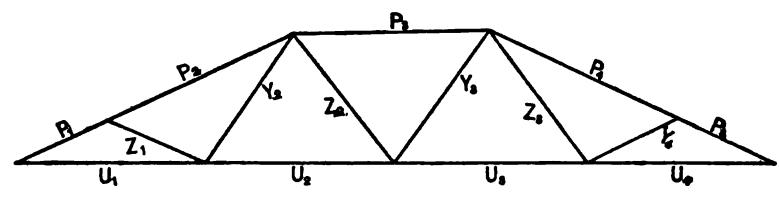
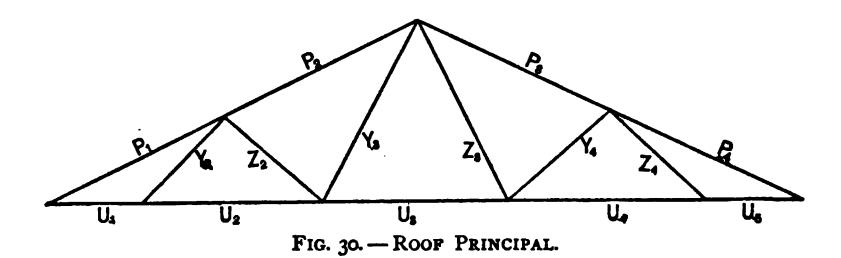


Fig. 29. — Roof Principal.



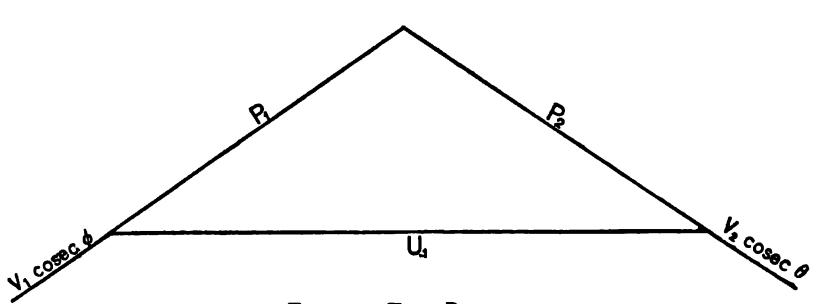


FIG. 31. — TIED RAFTERS.

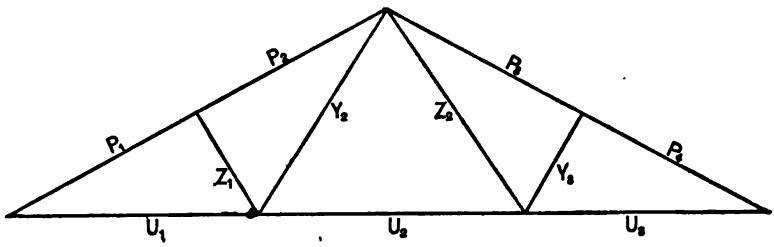


Fig. 32. — Roof Principal.

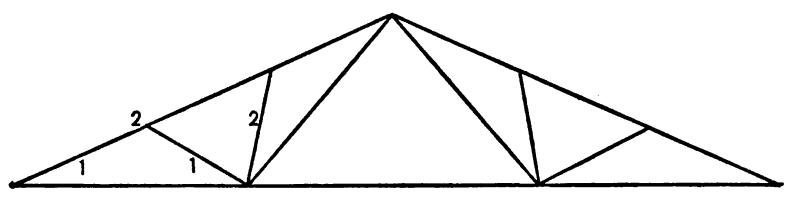


Fig. 33. — Roof Principal of Two Systems.

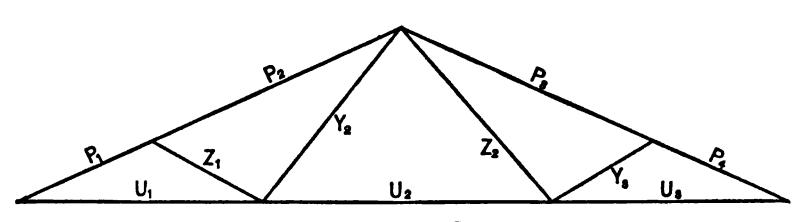


FIG. 33a. — FIRST SYSTEM.

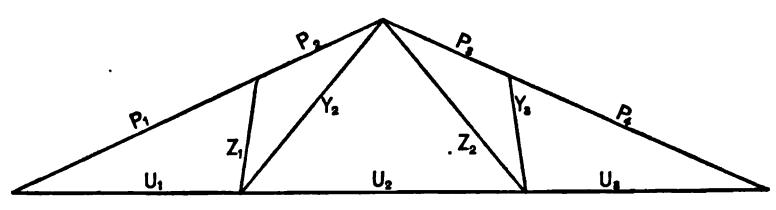


Fig. 336.—Second System.

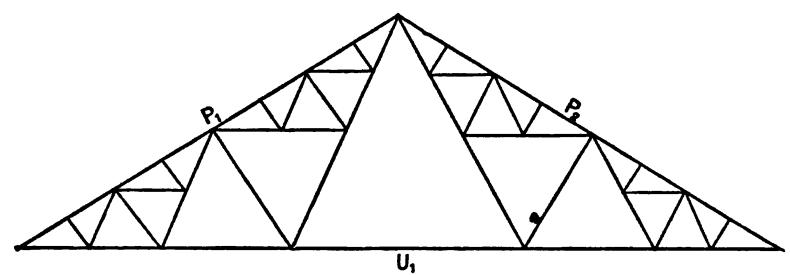


Fig. 34. — Roof Main, Compound System. (See Fig. 70, Class VIII.)

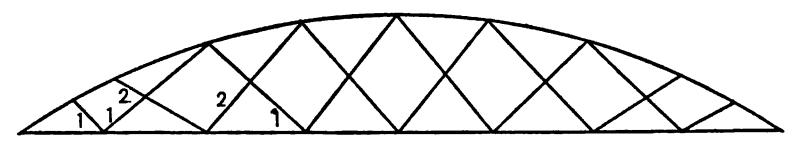


Fig. 35. — Bowstring of Two Systems. Triangular.

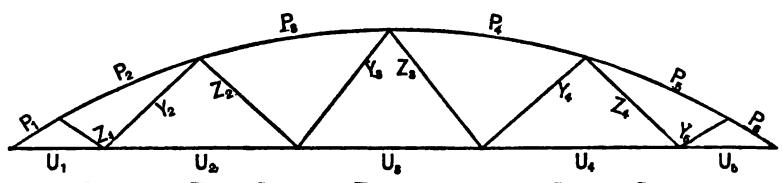


Fig. 35a. — First System. Fig. 27 shows the Second System.

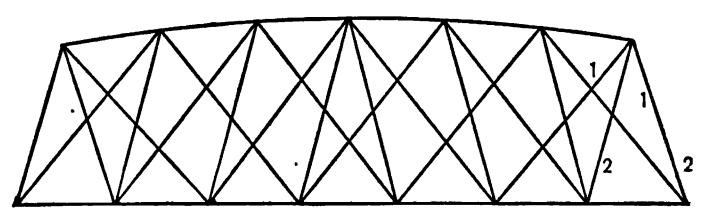


Fig. 36. — The Post Truss with Curved Top.

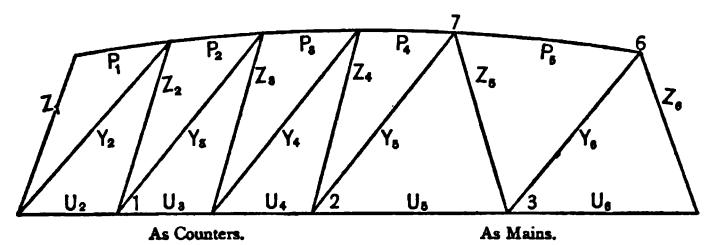


Fig. 36a. — First System.

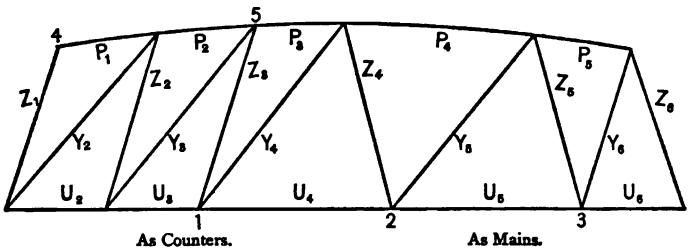


Fig. 36b. — Second System.

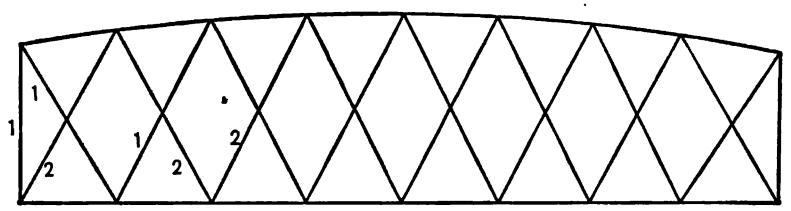


Fig. 37. — Double Triangular Truncated Bowstring. System of Kansas-City Bridge.

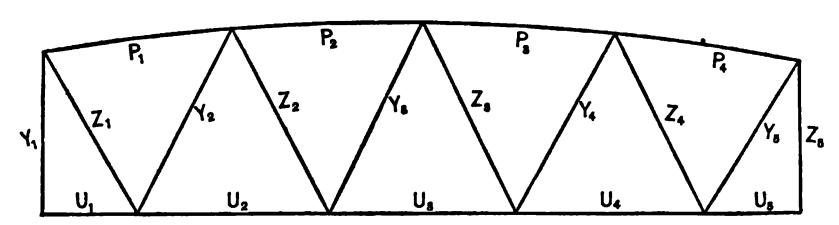


FIG. 37a. - FIRST SYSTEM.

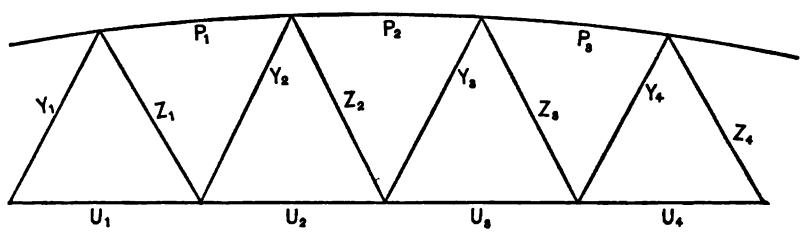


Fig. 376. - Second System.

## FORMULÆ FOR CLASS II. $\beta = 0$ .

Method of Moments.

$$H = M + h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = \frac{H_{r+1}}{\cos \alpha}.$$

$$U = -H_r.$$

$$Y = \frac{\Delta_r H}{\cos \phi}.$$

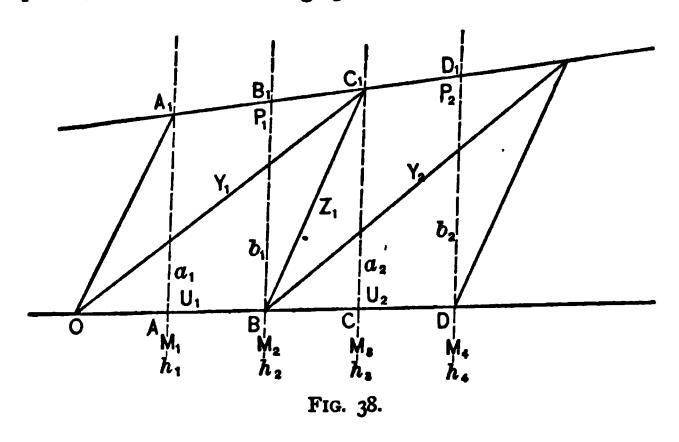
$$Z = \frac{\Delta_{r+1} H}{\cos \theta}.$$

In case of vertical end posts, as in Fig. 37a, where Y and Z become indeterminate by the above equations, we have

$$Y_1 = Z_1 \sin \theta_1 + P_1 \sin \alpha_1,$$
  $Z_5 = Y_5 \sin \phi_5 + P_4 \sin \alpha_4.$ 

Where a vertical section through any apex cuts a web member, as in Fig. 33b for  $Z_1 = Y_3$ , and in Figs. 36a and 36b for  $P_4$  and the counters Y and Z, we do not have H, the horizontal component of chord strain, equal to  $M \div h$ , but may proceed as follows:—

Find the moments  $M_1$ ,  $M_2$ ,  $M_3$ , etc., at vertical planes through consecutive apices. Call the heights above the bottom chord at which the cut diagonal meets these vertical planes in each panel, a and b, as in Fig. 38.



Then, taking moments about A and B, we have

$$M_1 = h_1 P_1 \cos \alpha_1 + \alpha_1 Y_1 \cos \phi_1, \qquad (145)$$

$$M_2 = h_2 P_1 \cos \alpha_1 + b_1 Y_1 \cos \phi_1, \qquad (146)$$

$$\therefore P_{1} = \frac{M_{2}a_{1} - M_{1}b_{1}}{(a_{1}h_{2} - b_{1}h_{1})\cos a_{1}}, \qquad (147)$$

$$Y_1 = \frac{M_1 h_2 - M_2 h_1}{(a_1 h_2 - b_1 h_1) \cos \phi_1}.$$
 (148)

Taking moments about  $A_1$  and  $B_2$ , there results

$$M_1 = -h_1 U_1 \cos \beta_1 - (h_1 - a_1) Y_1 \cos \phi_1,$$
 (149)

$$M_2 = -h_2 U_1 \cos \beta_1 - (h_2 - b_1) Y_1 \cos \phi_1,$$
 (150)

$$U_{i} = \frac{M_{2}(h_{1} - a_{1}) - M_{1}(h_{2} - b_{1})}{(a_{1}h_{2} - b_{1}h_{1})\cos\beta_{1}}.$$
 (151)

Similarly, or by increasing the indices of a, b,  $\alpha$ , and  $\phi$  by 1, and those of M and h by 2, we find

$$P_{2} = \frac{M_{4}a_{2} - M_{3}b_{2}}{(a_{2}h_{4} - b_{2}h_{3})\cos{\alpha_{2}}},$$

$$Y_{2} = \frac{M_{3}h_{4} - M_{4}h_{3}}{(a_{2}h_{4} - b_{2}h_{3})\cos{\phi_{2}}},$$

$$U_{2} = \frac{M_{4}(h_{3} - a_{2}) - M_{3}(h_{4} - b_{2})}{(a_{2}h_{4} - b_{2}h_{3})\cos{\beta_{2}}}.$$

Then, taking moments about C gives

$$M_{3} = h_{3}(P_{1}\cos\alpha_{1} + Y_{1}\cos\phi_{1} + Z_{1}\cos\theta_{1}) + a_{2}Y_{2}\cos\phi_{2},$$

$$\therefore Z_{1} = \left\{\frac{M_{3}}{h_{3}} - \frac{a_{2}}{h_{3}}Y_{2}\cos\phi_{2} - Y_{1}\cos\phi_{1} - P_{1}\cos\alpha_{1}\right\} \frac{1}{\cos\theta_{1}}.$$
 (152)

Now, in case of the Post Truss, we need only the counterstrains Y, since Z, P, and U have their greatest values as main strains.

And when both chords are horizontal, the horizontal projection of the Z member, or strut, is one-third of the horizontal projection of the Y member, or tie, as usually built; hence  $a = \frac{1}{2}b = \frac{1}{3}h$ ,

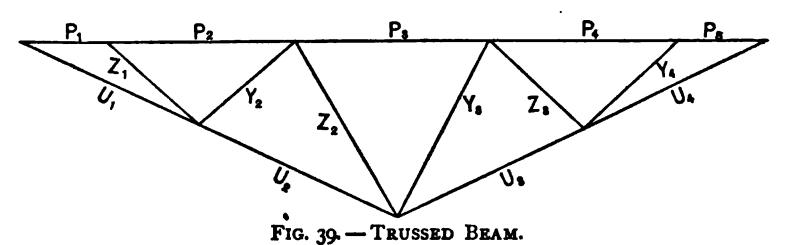
$$\therefore Y_1 = \frac{3(M_2 - M_1)}{h\cos\phi} = \frac{3\Delta_1 H}{\cos\phi}, \tag{153}$$

which, it will be seen, is the same thing as  $Y_r = \frac{\Delta H}{\cos \phi}$ , provided  $\Delta H$  is taken for the interval equal to the horizontal projection of the Y member, while  $\Delta_r H$  belongs to one-third of that interval; since the foremost end of the live load, at the instant the value of Y is here sought, is at the foot of the Y member, being applied either directly at the lower apex, or

indirectly at the upper apex, and reaching the bottom chord through the Z member terminating there. In other words, when the counter-strain Y, due live load, is greatest, there is no part of the live load applied on the right of the foot of the Y member, or of the top of the Z member above.

In multiple systems, where the chords are not straight lines, in finding total chord strains, care should be taken to reduce all strains that are to be added, so that their lines of action will be parallel; horizontal, for instance.

Class III. — Top Chord horizontal, other Members inclined.  $\alpha = 0$ .



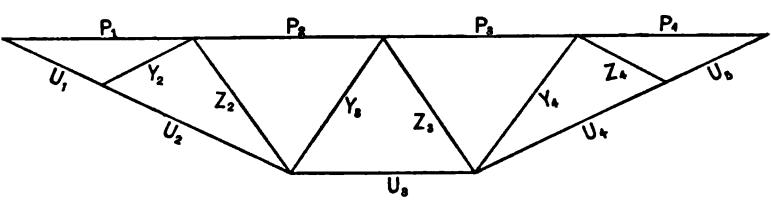


Fig. 40. — Trussed Beam.

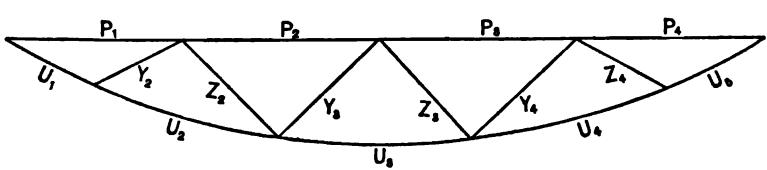


Fig. 41. - Invented Bowstring.

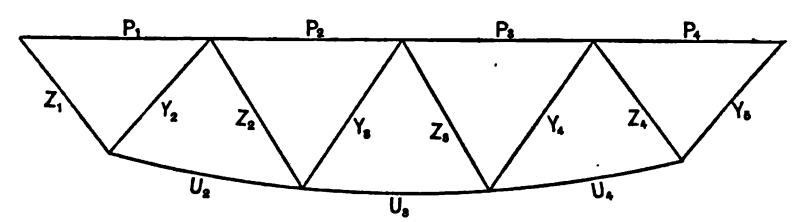


FIG. 42. - INVERTED TRUNCATED BOWSTRING.

FORMULÆ FOR CLASS III.  $\alpha = 0$ .

Method of Moments.

$$H = M + h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_{r+1}.$$

$$U = -H_r + \cos \beta.$$

$$Y = \Delta_r H + \cos \phi.$$

$$Z = \Delta_{r+1} H + \cos \theta.$$

Class IV. — Both Chords horizontal, Web Members inclined.  $\alpha = 0$ ,  $\beta = 0$ .

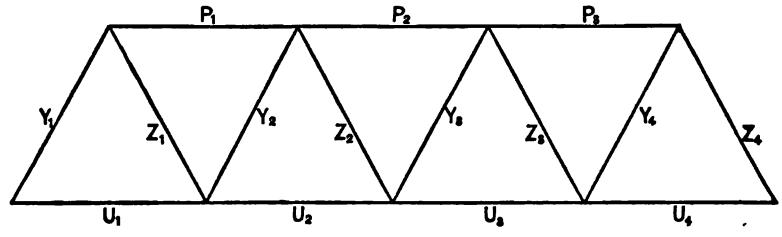


Fig. 43.—The Triangular Girder. Erect, or "Through."

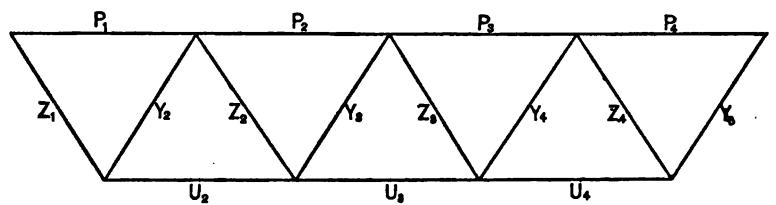


Fig. 44. — Triangular Girder. Suspended, or "Deck." a = 0,  $\beta = 0$ .

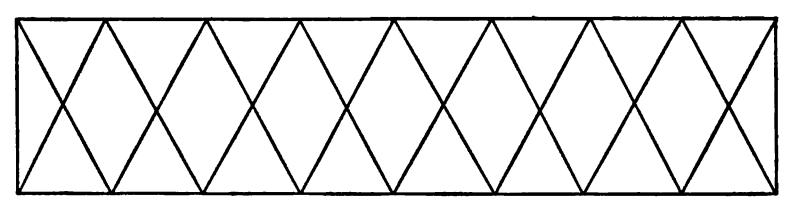


Fig. 45.—Double Triangular Girder. Figs. 43 and 44 combined.

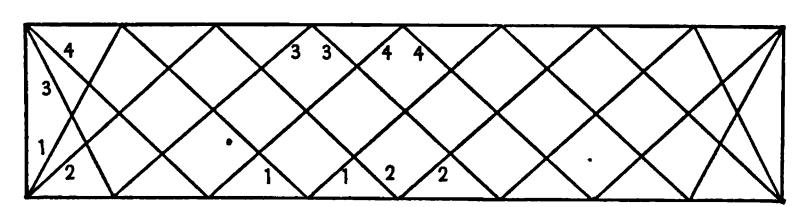


Fig. 46. — Quadruple Triangular System. Each System independent.

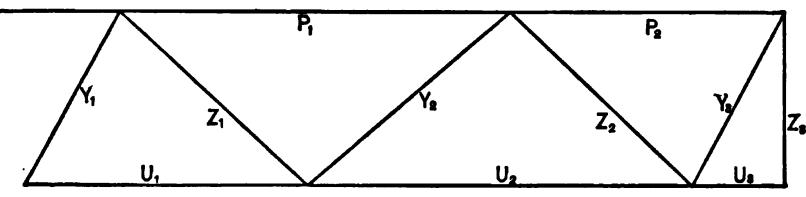


FIG. 46a. - FIRST SYSTEM.

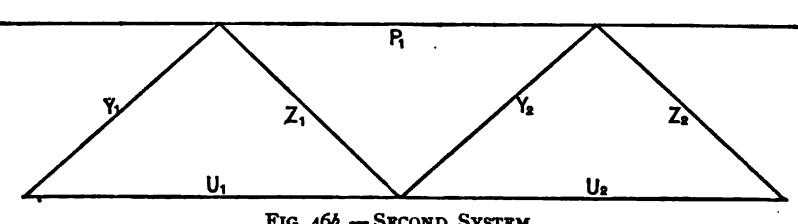


Fig. 46b. - Second System.

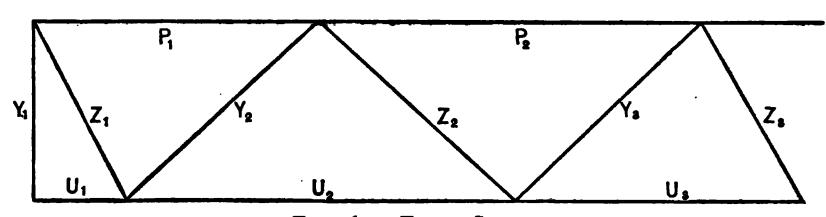


Fig. 46c. — Third System.

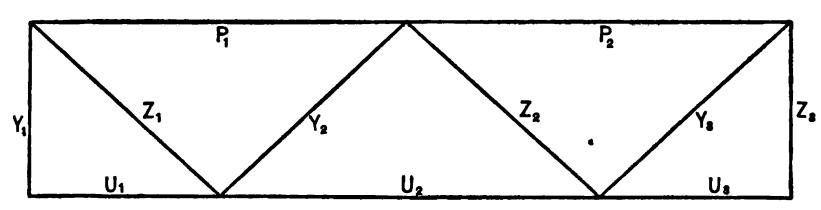


Fig. 46d. — Fourth System.

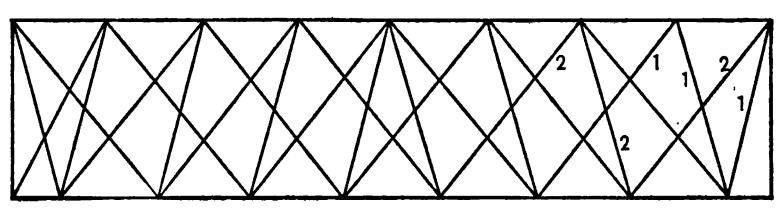


Fig. 47.—The Post Truss. Two Systems.

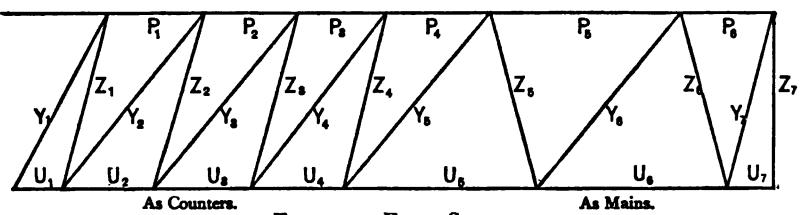


FIG. 47a. — FIRST SYSTEM.

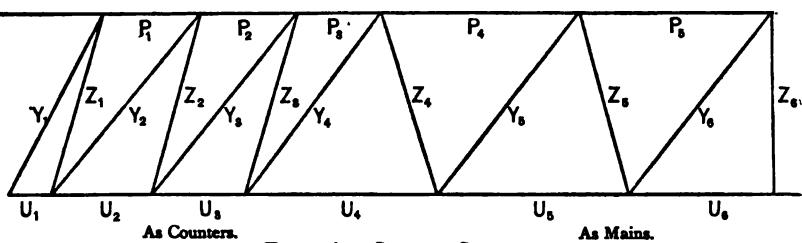


FIG. 47b. — SECOND SYSTEM.

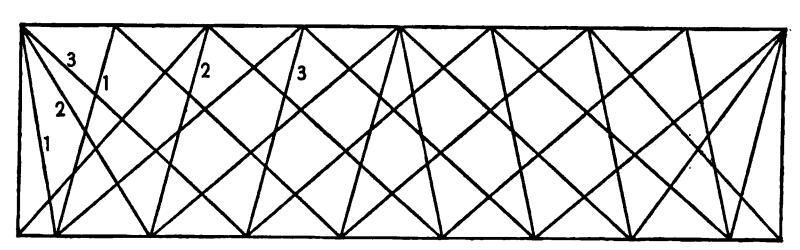


FIG. 47c. - POST TRUSS. THREE SYSTEMS.

Formulæ for Class IV.  $\alpha = 0$ , = 0.

Method of Moments.

$$H = M + h.$$

$$\Delta H = \Delta M + h.$$

$$P = H_{r+1}.$$

$$U = -H_r.$$

$$Y = \Delta_r H \div \cos \phi.$$

$$Z = \Delta_{r+1}H + \cos\theta.$$

## Class V. — Alternate Web Members vertical. Both Chords inclined.

Generally 
$$\left\{ \begin{array}{l} \text{Verticals in compression,} \\ \text{Diagonals in tension.} \end{array} \right\} \theta = 90^{\circ}.$$

Only one set of diagonals shown in figures. These are counters in first half-span, mains in second half-span.

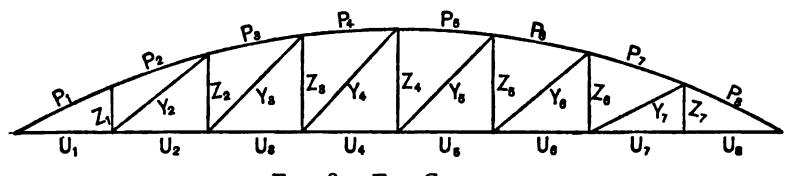


FIG. 48. — THE CRESCENT.

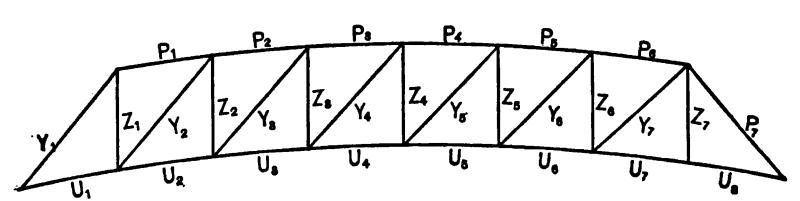


Fig. 49. — Truncated Crescent.

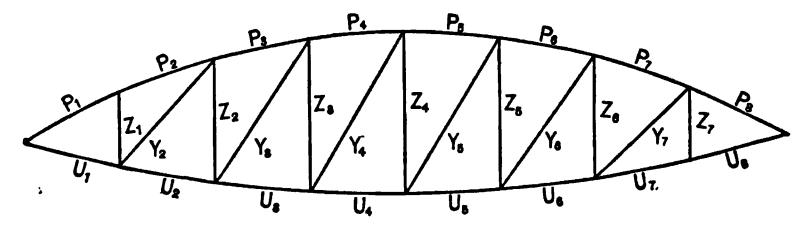


FIG. 50. — DOUBLE BOW, OR BRUNEL GIRDER.

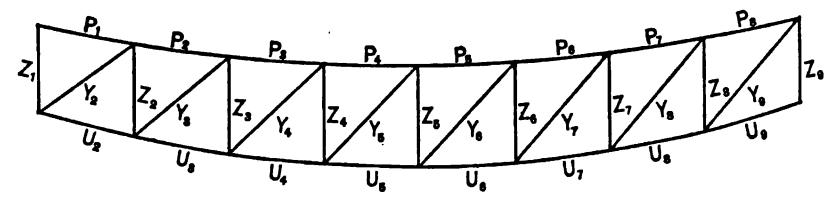


FIG. 51. - TRUNCATED CRESCENT INVERTED.

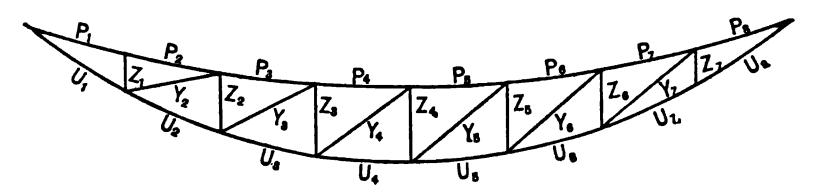


FIG. 52. — CRESCENT SUSPENDED.

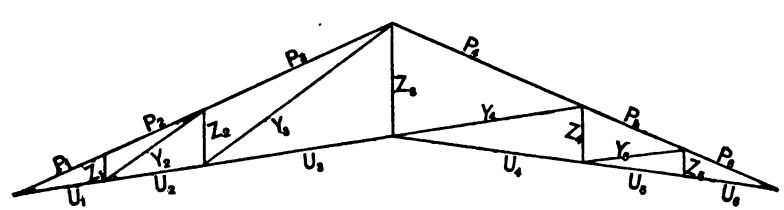


Fig. 53.—Roof Principal.

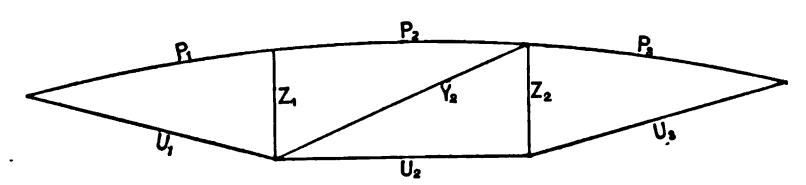


FIG. 54. — TRUSSED RIB.,

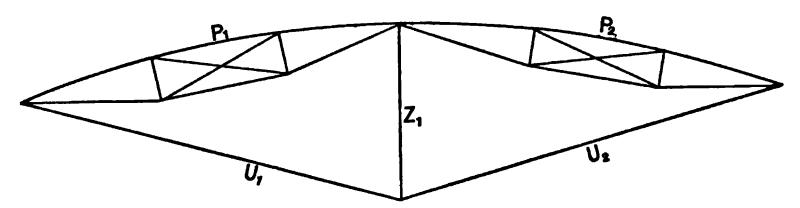


Fig. 55.—Dome Principal. Primary System. Secondary System same as in Fig. 54.

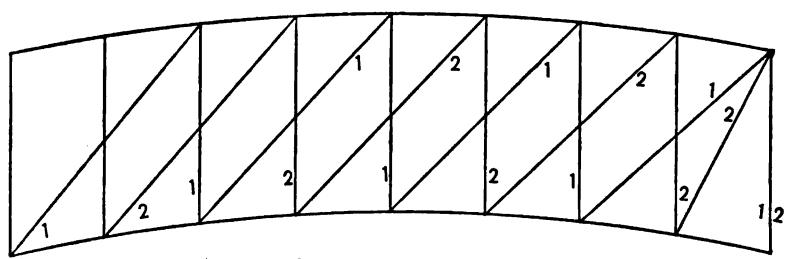


Fig. 56. — Bent Truss. Double System.

Formulæ for Class V.  $\theta = 90^{\circ}$ .

Method of Moments.

$$H = M + h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H + \cos \alpha.$$

$$U_{-1} = -H + \cos \beta.$$

$$Z = P_{r+1} \sin \alpha_{r+1} - P_r \sin \alpha_r - Y_r \sin \phi_r$$

$$= H_{r+1} \tan \alpha_{r+1} - H_r \tan \alpha_r - \Delta_r H \tan \phi_r$$

 $Y = \Delta_r H + \cos \phi.$ 

$$Z = U_r \sin \beta_r - U_{r+1} \sin \beta_{r+1} - Y_r \sin \phi_r$$

$$= -H_r \tan \beta_r + H_{r+1} \tan \beta_{r+1} - \Delta_r H \tan \phi_r.$$
(Load applied at top.)

(Load applied at bottom.)

The value of Z in equation

$$Z = \Delta H + \cos \theta,$$

here becomes indeterminate, since  $\Delta H = 0$  for the horizontal projection of the Z member, and  $\cos 90^{\circ} = 0$ .

Class VI. — Both Chords inclined. Alternate Web Members vertical.

Generally 
$$\left\{ \begin{array}{l} \text{Verticals in tension,} \\ \text{Diagonals in compression.} \end{array} \right\} \phi = 90^{\circ}.$$

Only one set of diagonals shown in figures. These are counters in first half-span, mains in second half-span.

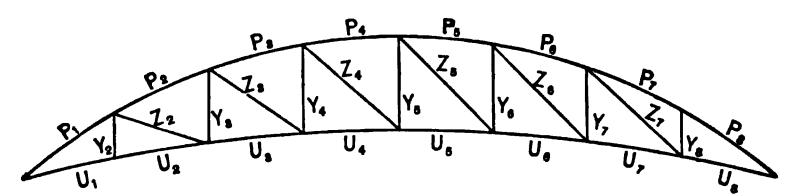


FIG. 57. — THE CRESCENT.

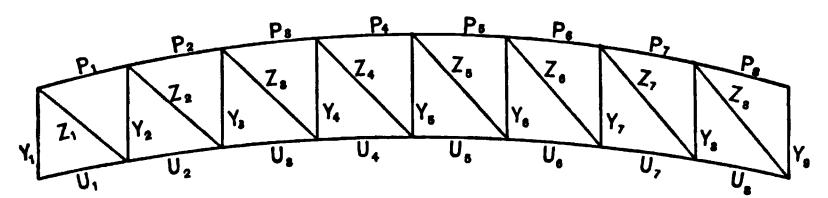


Fig. 58. — Truncated Crescent.

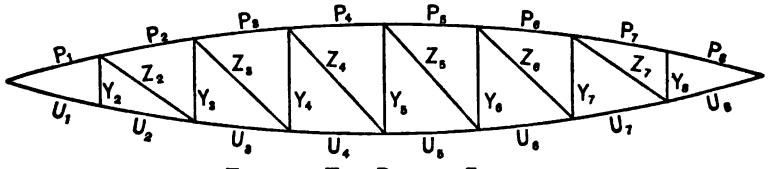


FIG. 59. — THE BRUNEL GIRDER.

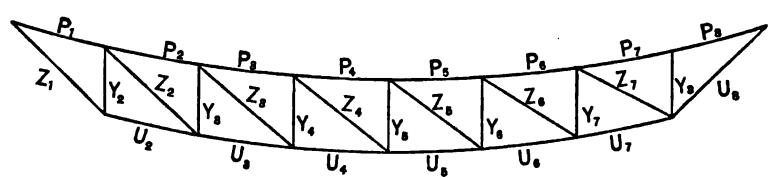


Fig. 60. — Truncated Crescent suspended.

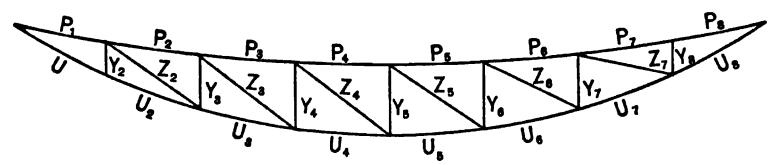


FIG. 61. — SUSPENDED CRESCENT.

Formulæ for Class VI.  $\phi = 90^{\circ}$ .

Method of Moments.

$$H = M + h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H + \cos \alpha.$$

$$U_{-1} = -H + \cos \beta.$$

$$Y = P_{r+1} \sin \alpha_{r+1} - P_r \sin \alpha_r - Z_{r+1} \sin \theta_{r+1}$$

$$= H_{r+1} \tan \alpha_{r+1} - H_r \tan \alpha_r - \Delta_{r+1} H \tan \theta_{r+1}.$$
(Load applied at bottom.)

$$Y = U_r \sin \beta_r - U_{r+1} \sin \beta_{r+1} - Z_r \sin \theta_r$$

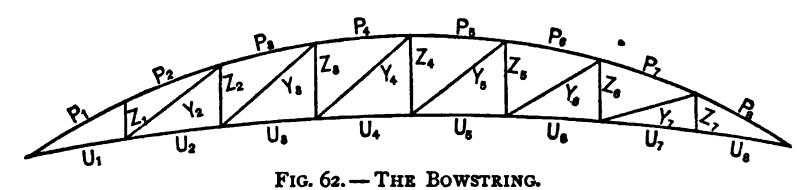
$$= -H_r \tan \beta_r + H_{r+1} \tan \beta_{r+1} - \Delta_r H \tan \theta_r.$$
(Load applied at top.)
$$Z = \Delta_r H + \cos \theta.$$

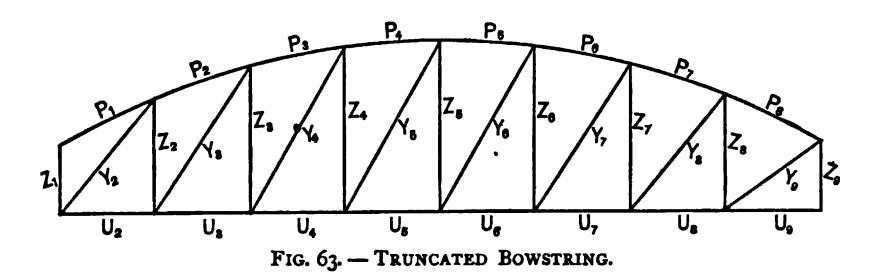
Multiple systems of this class are seldom built, since long struts are not economical.

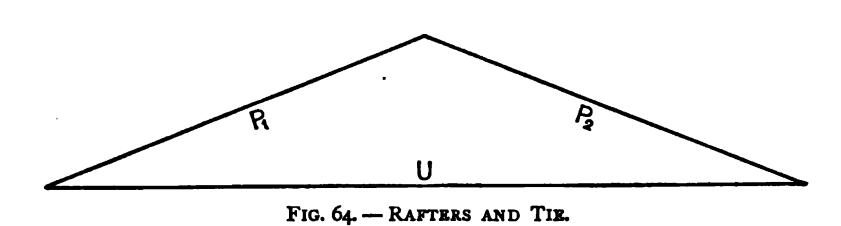
Class VII. — Bottom Chord Horizontal.  $\beta = 0$ . Alternate Web Members vertical.  $\theta = 90^{\circ}$ .

In general, { Verticals in compression, Diagonals in tension.

But one set of diagonals shown in figures. These are counters in first half-span, mains in second half-span.







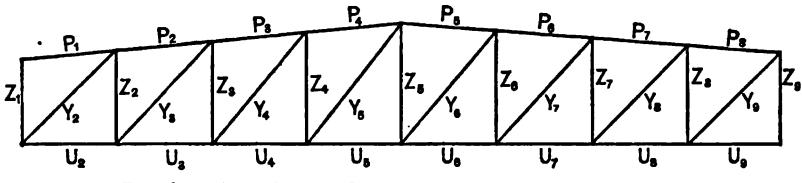
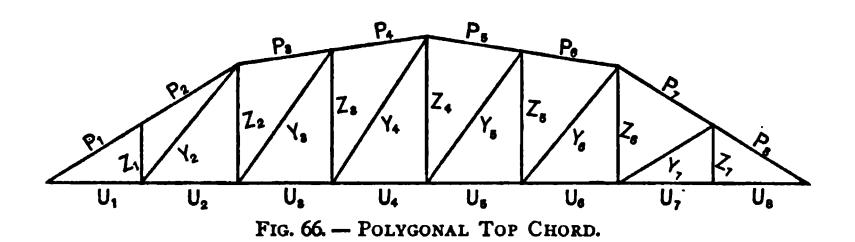
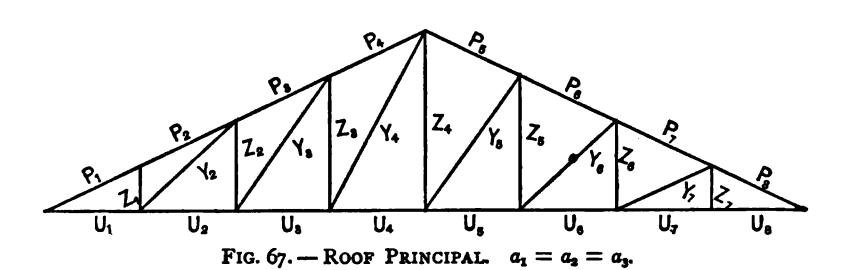
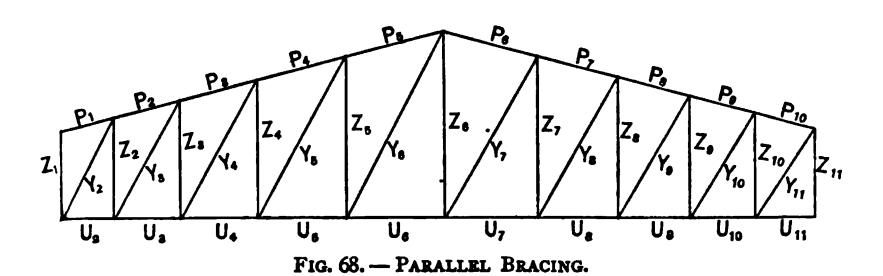


Fig. 65. — Top Chord Uniformly sloped.  $a_1 = a_2 = a_3$ .







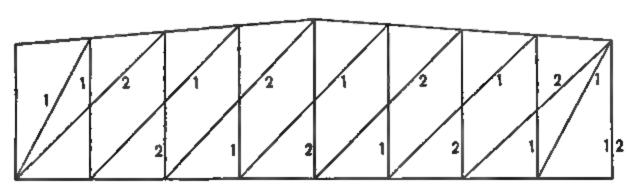


Fig. 69. - Uniform Slope. Double System.

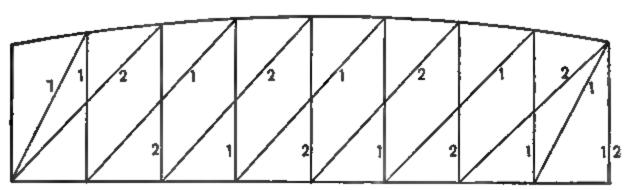


Fig. 70. — CURVED TOP. DOUBLE SYSTEM.





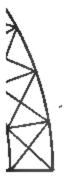


Fig. 71. - ROOF PRINCIPAL. (SEE Fig. 23.)

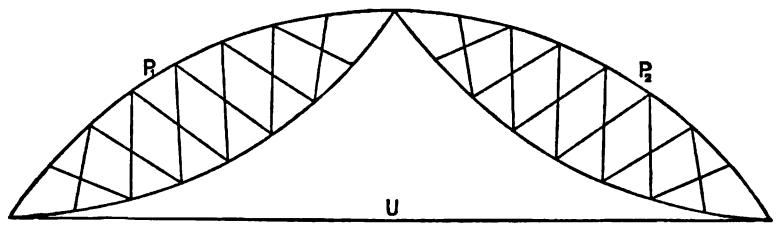


FIG. 71a. — THE TWIN FISHES, LONG SPAN. (SEE FIG. 22.)

Formulæ for Class VII.  $\beta = 0$ ,  $\theta = 90^{\circ}$ .

#### Method of Moments.

$$H = M + h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_r + \cos \alpha.$$

$$U = -H_{r+1}.$$

$$Y = \Delta_r H + \cos \phi_r.$$

$$Z = \Delta_{r+1} H \tan \phi_{r+1}.$$
simultaneous.

Foremost end of live load at  $Z_{r-1}$  for maximum Y and  $Z_r$ .

When it is desired to have the diagonals in each half-span parallel for a given number of panels, as in Fig. 68, the lengths of the panels and the inclination of the diagonals may be found as follows:—

Let l = length of span.

 $h_o$  = height at each end.

h = central height.

m = number of panels in each half-span.

 $\phi$  = inclination to horizon of counters in first half-span, and of mains in second half-span.

 $\alpha$  = inclination of top chord.

 $\Delta l =$  the variable panel length.

Then

$$\Delta l = \frac{h_r}{\tan \phi}$$
, and  $\Delta h = h_r \frac{\tan \alpha}{\tan \phi}$ .

 $\tan \alpha = \frac{h - h_o}{\frac{1}{2}l}$ .

$$h_{m-1} = h - \Delta h = h - h \frac{\tan \alpha}{\tan \phi} = h \left( 1 - \frac{\tan \alpha}{\tan \phi} \right).$$

$$h_{m-2} = h_{m-1} \left( 1 - \frac{\tan \alpha}{\tan \phi} \right) = h \left( 1 - \frac{\tan \alpha}{\tan \phi} \right)^2.$$

 $h_o = h \left( 1 - \frac{\tan \alpha}{\tan \phi} \right)^m, \qquad (154)$ 

$$\therefore \tan \phi = \frac{\tan \alpha}{1 - \left(\frac{h_0}{h}\right)^{\frac{1}{m}}}$$
 (155)

for the first half-span.

Similarly, for the second half-span

$$h = h_o \left(1 + \frac{\tan \alpha}{\tan \phi}\right)^m, \qquad (156)$$

$$\tan \phi = \frac{\tan \alpha}{1 + \left(\frac{h}{h_o}\right)^{\frac{1}{m}}}.$$
 (157)

Generally,

$$h_r = h_0 \left( 1 + \frac{\tan \alpha}{\tan \phi} \right)^r. \tag{158}$$

Class VIII. — Top Chord horizontal.  $\alpha = 0$ . Struts vertical.  $\theta = 90^{\circ}$ .

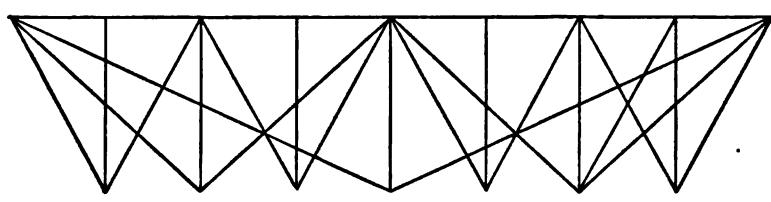


Fig. 72. — The Fink Truss.

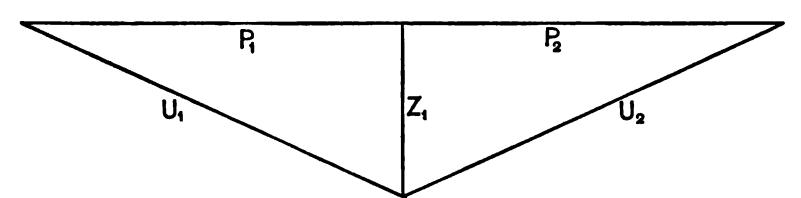


Fig. 72a. — Main Suspenders.

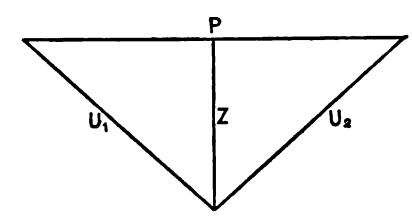
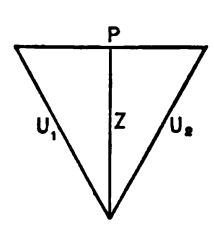


Fig. 726. — Secondaries.



TERTIARIES.

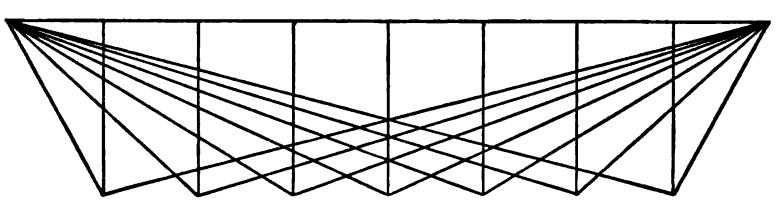


Fig. 73. — THE BOLLMAN TRUSS.

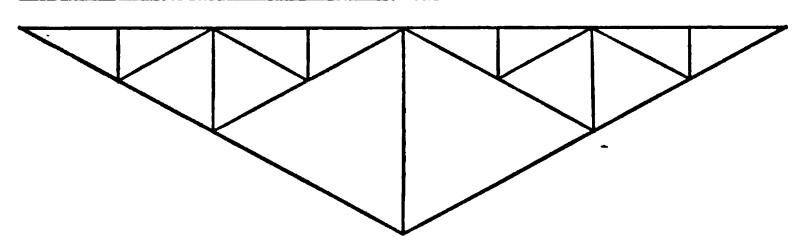


Fig. 74. — Trussed Rafter of Fig. 34.

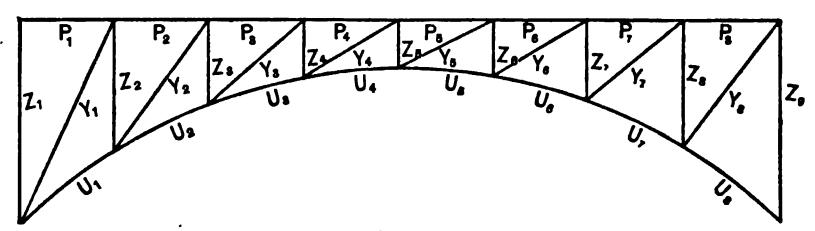


Fig. 75. — Spandrel filled.

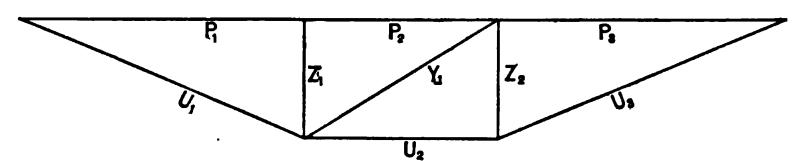


Fig. 76. — Trussed Beam, Three Links.

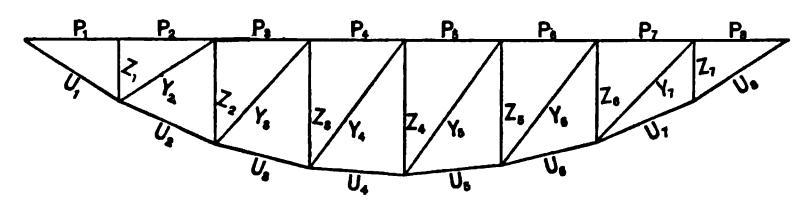


Fig. 77. — Catenarian Links.

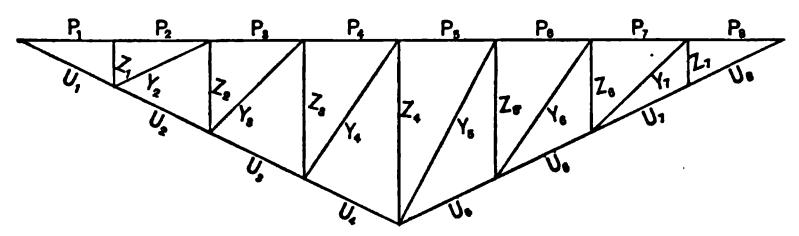


Fig. 78. — Trussed Bram.

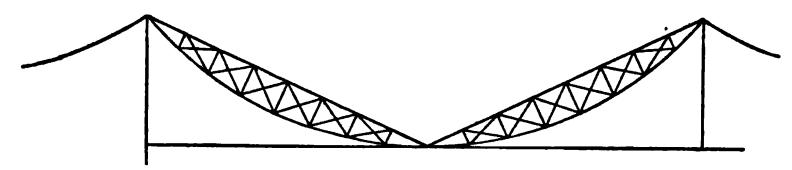


Fig. 79. — The Point Suspension, Stiffened Catenary.

Formulæ for Class VIII.  $\alpha = 0$ ,  $\theta = 90^{\circ}$ .

Method of Moments.

$$H = M + h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_{r+1}.$$

$$U = -H_r + \cos \beta.$$

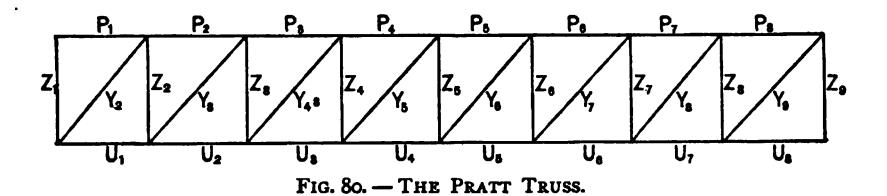
$$Y = \Delta H + \cos \phi.$$

$$Z = \Delta H \tan \phi.$$

When the vertical member has no diagonal attached at its top, then, of course, the strain upon the vertical is, for Class VIII., equal to the load applied at the upper apex.

### Class IX. — Both Chords horizontal. $\alpha = 0$ .

Verticals in compression.  $\beta = 0$ . Diagonals in tension.  $\theta = 90^{\circ}$ .



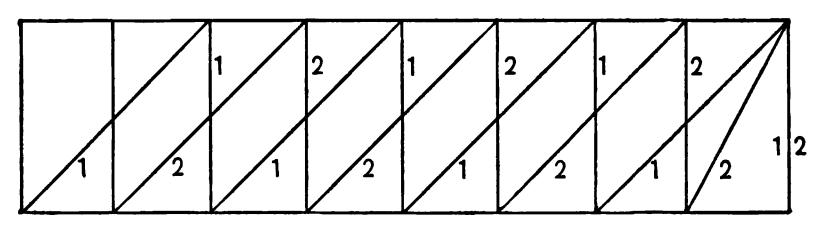


Fig. 80a. — The Linville, or Pratt of Two Systems.

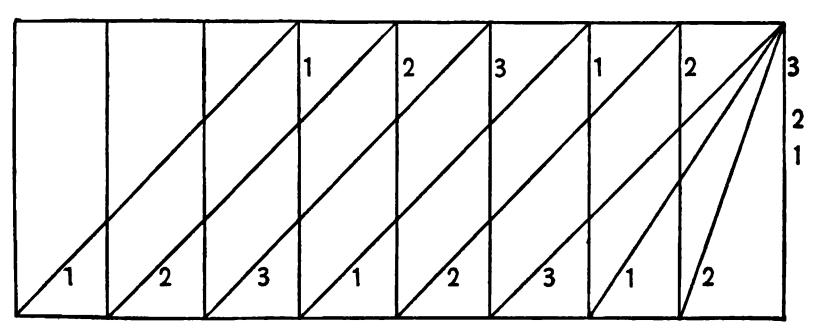
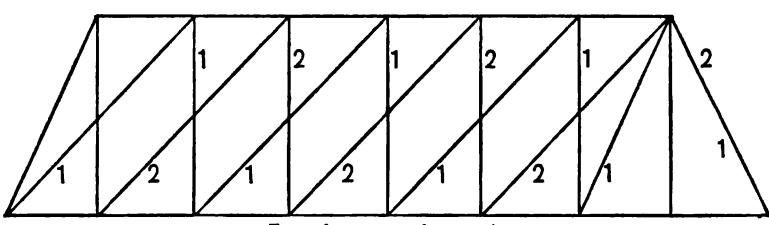


Fig. 800. — Pratt Truss of Three Systems.



For end struts,  $\tan \theta = 2 \tan \phi$ .

Fig. 80c. — Linville, with Inclined End-Posts.

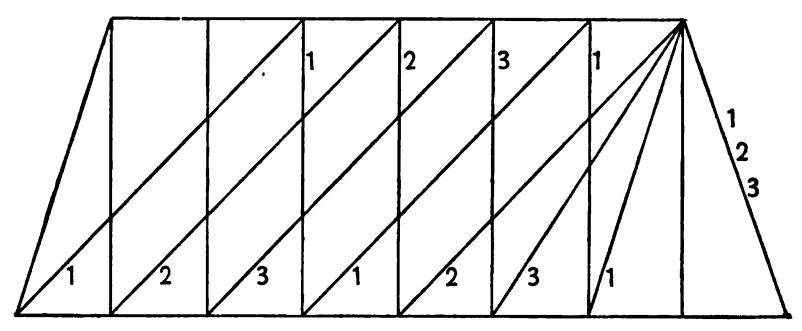


Fig. 8od. — Three Systems, Inclined End Posts.

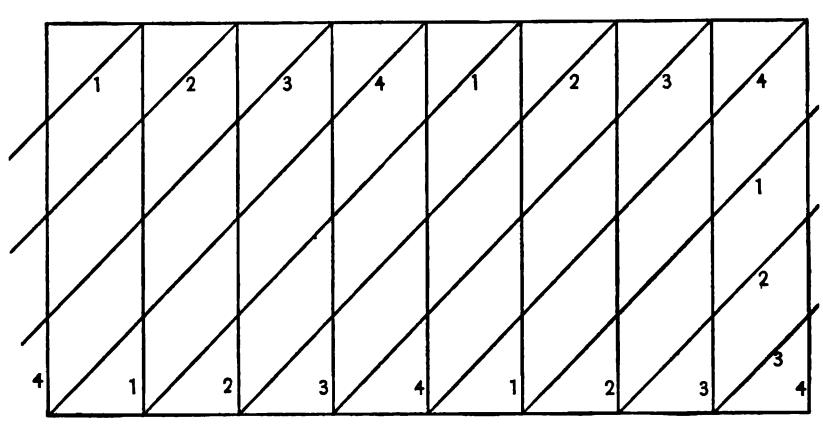


Fig. 80% — Truss Systems of Niagara Bridge.

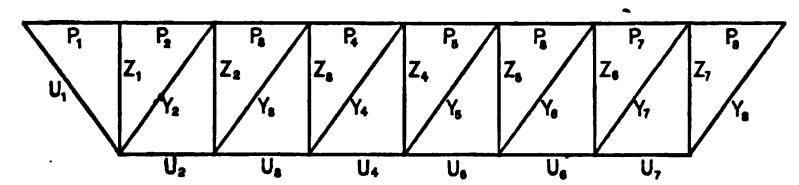


Fig. 80f. — Pratt Truss suspended.

Formulæ for Class IX.  $\alpha = 0$ ,  $\beta = 0$ ,  $\theta = 90^{\circ}$ .

Method of					
Moments.	Moments and Shearing-Forces.				
$H = M + h.$ $\Delta H = \Delta M + h.$ $P = H_r.$ $U = -H_{r+1}.$ $Y = \Delta H + \cos \phi.$ $Z = \Delta H \tan \phi.$	$H = (M_W + M_L) + h.$ $S = S_W + S_L.$ $P = H_{W+L}.$ $U = -H_{+1W+L}.$ $Y = -S + \sin \phi.$ $Z = S_{+1}, \text{ load applied at top.}$ $Z = S, \text{ load applied at bottom.}$				

## Class X. — Bottom Chord Horizontal. $\beta = 0$ .

Verticals in tension.  $\phi = 90^{\circ}$ . Diagonals in compression.

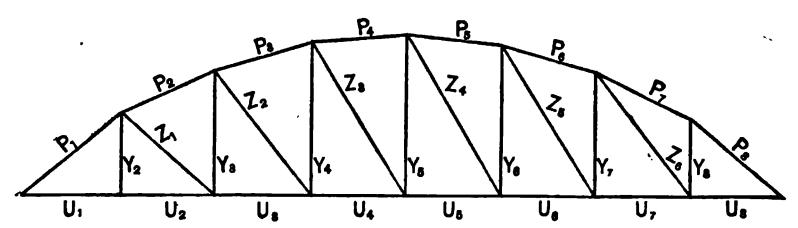


Fig. 81. — The Polygonal Bowstring.

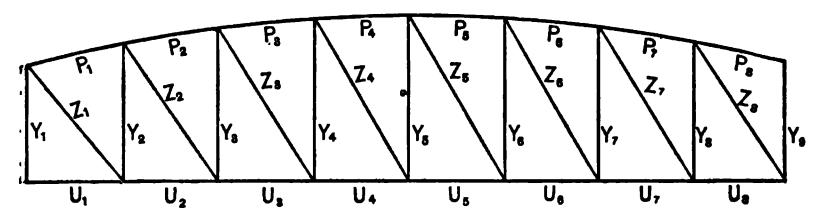


Fig. 82. — The Howe Truss, with Curved Top.

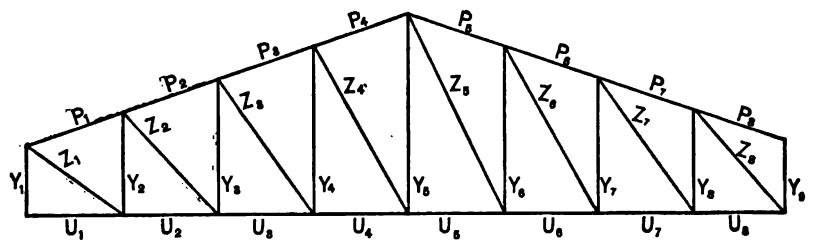


Fig. 83. — Howe Truss, Inclined Top Chord.

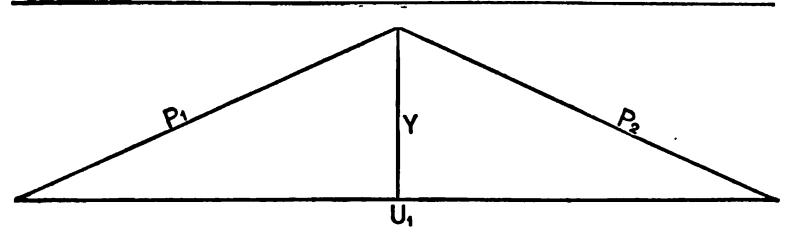


Fig. 84. — RAFTERS, WITH VERTICAL TIE.

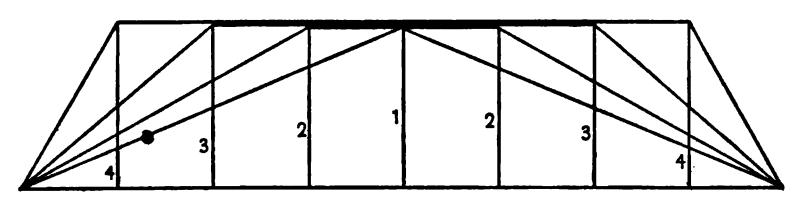


Fig. 84a. — Systems of the Schaffhäusen Truss.

Formulæ for Class X.  $\beta = 0$ ,  $\phi = 90^{\circ}$ .

Method of Moments.

$$H = M + h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_{r+1} + \cos \alpha.$$

$$U = -H_r.$$

$$Y = \Delta H \tan \theta.$$

$$Z = \Delta H + \cos \theta.$$

Class XI. — Top Chord horizontal.  $\alpha = 0$ . Struts inclined. Ties vertical.  $\phi = 90^{\circ}$ .

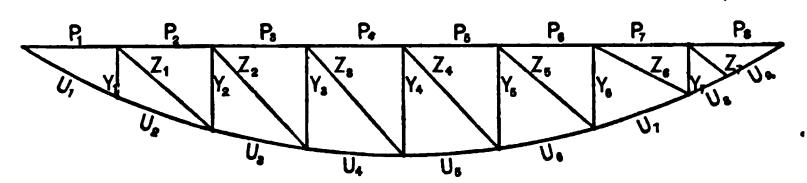


Fig. 85. — Suspended Bow.

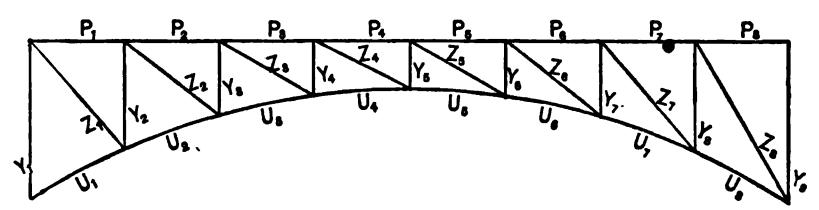


FIG. 86. — FILLED SPANDRELS.

Formulæ for Class XI.  $\alpha = 0$ ,  $\phi = 90^{\circ}$ .

Method of Moments.

$$H = M + h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_{r+1}.$$

$$U = -H_r + \cos \beta_r.$$

$$Y = \Delta H \tan \theta.$$

$$Z = \Delta H + \cos \theta.$$

Class XII. — Both Chords horizontal.  $\alpha = 0$ ,  $\beta = 0$ . Struts inclined. Ties vertical.  $\phi = 90^{\circ}$ .

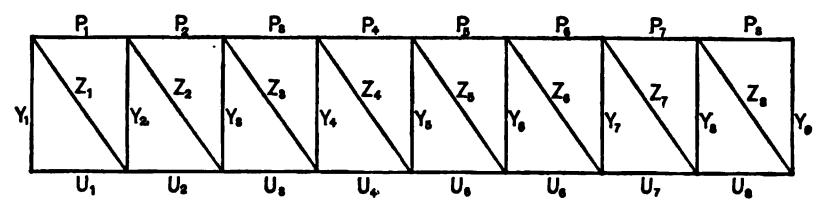


Fig. 87. — The Howe Truss.

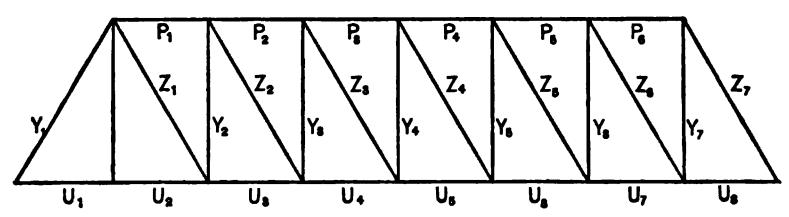


Fig. 88. — Howe Truss, Inclined End Posts.

Formulæ for Class XII.  $\alpha = 0$ ,  $\beta = 0$ ,  $\phi = 90^{\circ}$ .

#### Method of Moments and Shearing-Forces. Moments. $H = (M_W + M_L) + h.$ H = M + h. $\Delta H = \Delta M + h.$ $S = S_W + S_L$ . $P = H_{+1W+L}.$ $=H_{r+1}$ $=-H_r$ $U = -H_{W+L}.$ $\boldsymbol{U}$ $Y = -S_{+}$ $= \Delta H \tan \theta$ . $\boldsymbol{Y}$ $Z = \Delta H + \cos \theta$ . $Z = S + \sin \theta$ .

NAL FORCES OF A BEAM NUOUS WEB,

trticular Cross-Sections of Web.

moment of resistance offered by the cohesion of the particles of the material composing a beam, we now proceed to illustrate.

Let ABCD, Fig. 89, be the vertical longitudinal central section of a beam of any cross-section whatever, under the influence of given applied forces or pressures.

It is required to find the moment of resistance offered by the material of the

Let NS be the intersection of the neutral surface of the beam with the plane of the paper. The neutral surface of a beam coincides with the position of that longitudinal lamina which, for a given strain, is neither compressed nor elongated.

All fibres not in the neutral surface are assumed to be increased or diminished in length by a quantity in direct proportion to their distance from the neutral surface, and also in direct proportion to the intensity of the force acting on the fibres.

Let f = the force acting on a unit of area of any given normal section, at right angles to the section, and at the unit's distance from the neutral surface, either above or below.

dz = an element of the thickness of the beam.

dy = an element of its depth.

y = the distance of the fibre whose area is dz dy from the neutral surface.

Then the pressure upon the area dz dy is

and the elementary moment due this stress is

$$dM = f y^2 dy dz;$$

whence the total moment for the cross-section is

$$R = M = f \iint y^2 \, dy \, dz, \qquad (159)$$

which expression is to be integrated between limits depending on the form of the given cross-section whose centre of gravity may be taken as the origin of co-ordinates.

51. We will now apply (159) to the determination of the moment of resistance, R, for various cross-sections occurring in practice.

Let the beam have a rectangular cross-section of the breadth b, and the height h, as in Fig. 90.

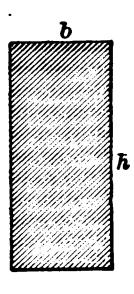


FIG. 90.

Then equation (159) becomes

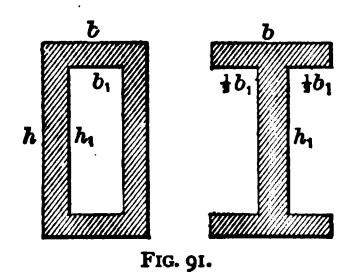
$$R = f \int_{-\frac{1}{4}b}^{+\frac{1}{4}b} \int_{-\frac{1}{4}h}^{+\frac{1}{4}h} dy \, ds = fb \int_{-\frac{1}{4}h}^{+\frac{1}{4}h} dy,$$

$$\therefore R = \frac{1}{12} fbh^3 = \frac{1}{8} Bbh^2, \tag{160}$$

where f is the unit strain at the unit's distance from the neutral surface, and  $B = \frac{1}{2}hf$  = the unit strain at the distance  $\frac{1}{2}h$  from the neutral surface, or at the upper and lower surfaces of the beam, since the neutral surface is here assumed to be in its centre. The quantity B is the unit strain which, at the instant of rupture, would be developed at the upper and lower surfaces of a beam having its neutral surface midway between those outer surfaces. B is called the *modulus of rupture*, or the ultimate unit resistance of the material to cross-breaking.

A table giving values of B is inserted in article 60.

# 52. Beam of Hollow Rectangular Section, or Beam of Equal Flanges, Fig. 91.



Let h = height of beam.

 $h_r$  = height of cavity or web.

b = breadth of beam or flange.

 $b_i$  = breadth of cavity.

Then, as in article 51, we shall have, for the whole area  $b \times h$ ,

$$R_1 = \frac{1}{12} fbh^3;$$

and for the area of the cavity  $b_1 \times h_2$ , or  $2 \times \frac{1}{2}b_1 \times h_2$ ,

$$R_2 = \frac{1}{12} f b_1 h_1^3.$$

Whence, for net area of cross-section,

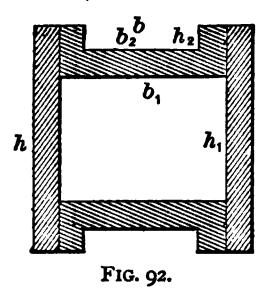
$$R = \frac{1}{12}f(bh^3 - b_1h_1^3) = \frac{1}{6}B\frac{bh^3 - b_1h_1^3}{h}, \qquad (161)$$

where  $B = \frac{1}{2}hf$  = unit strain at the upper and lower surfaces of the beam.

If the beam is square and hollow, so that h = b, and  $h_1 = b_1$ , we have, from equation (161),

$$R = \frac{1}{8}B\frac{h^4 - h_r^4}{h}.$$
 (162)

## 53. Beam composed of Two Vertical Plates and Two Horizontal Channels.



Let the two plates and the two channels, Fig. 92, have equal cross-sections.

b = entire breadth of beam.

h = entire height of beam.

 $b_{\rm r}$  = width of channel.

 $b_2$  = width of its web.

 $h_{\rm r}$  = distance between channels.

 $h_2$  = depth of channel cavity.

The neutral axis (that is, the line of intersection of the neutral surface with the normal section) is here central.

Whence, as in article 51, we have, for the area  $(b - b_2) \times h$ ,

$$R_1 = \frac{1}{12}f(b - b_2)h^3;$$

for the area  $b_2 \times (h - 2h_2)$ ,

$$R_2 = \frac{1}{12} f b_2 (h - 2h_2)^3;$$

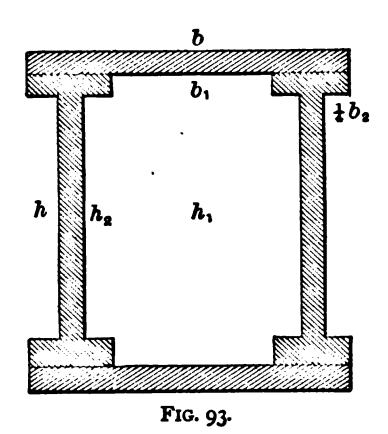
for the area  $b_1 \times h_2$ ,

$$R_3 = \frac{1}{12} f \delta_1 h_1^3.$$

Whence the total moment of resistance,

$$R = \frac{B}{6h} [(b - b_2)h^3 + b_2(h - 2h_2)^3 - b_1h_1^3], \quad (163)$$

where  $B = \frac{1}{2}hf = u$  strain in extreme top and bottom fibres. 54. Beam composed of Two Vertical I-Beams and Two Equal Horizontal Plates.



In Fig. 93, let h = height of beam.

 $h_1 = \text{height of } \mathbf{I}\text{-beams.}$ 

 $h_2$  = height of their webs.

b =width of plates.

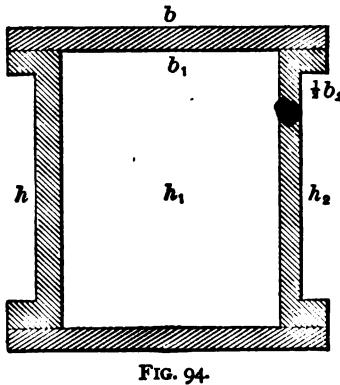
 $b_1$  = width between beams.

 $b_2$  = width of cavities of beams.

Then, proceeding as in the last article, we find

$$R = \frac{B}{6h}(bh^3 - b_1h_1^3 - b_2h_2^3). \tag{164}$$

55. Beam composed of Two Vertical Channels and Two Horizontal Plates.



In Fig. 94, let h = height of beam.

 $h_{\rm r}$  = height of channels.

 $h_2$  = height of their webs.

b =width of plates.

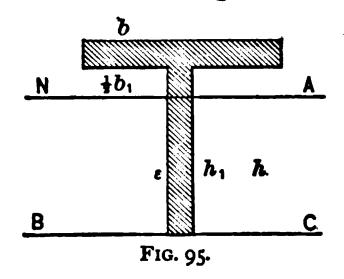
 $b_{\rm r}$  = width between channels.

 $b_2$  = width of cavities.

Then we have

$$R = \frac{B}{6h}(bh^3 - b_1h_1^3 - b_2h_2^3). \tag{165}$$

56. Beam with but One Flange.



Let the cross-section, Fig. 95, have the form of the letter T.

= width of flange.

= whole height of beam.

 $b - b_1 =$  thickness of web.

= height of web.

= distance of neutral axis NA from any line, BC, parallel to NA, in the plane of the given crosssection.

1st, To find e, and determine the position of the neutral axis.

Take the moment of the area of the section about BC as an axis, and divide this moment by the area. The quotient will be the value of a.

Moment of flange about BC

$$= b(h - h_1) \times \frac{1}{2}(h + h_1) = \frac{1}{4}b(h^2 - h_1^2).$$

Moment of web about  $BC = h_i(b - b_i) \times \frac{1}{2}h_i = \frac{1}{2}h_i^2(b - b_i)$ .

Total moment of area about BC is

$$\frac{1}{2}\delta(h^2-h_1^2)+\frac{1}{2}h_1^2(\delta-\delta_1)=\frac{1}{2}(\delta h^2-\delta_1 h_1^2).$$

Total area =  $bh - b_1h_1$ ; therefore

$$= \frac{bh^2 - b_1h_1^2}{2(bh - b_1h_1)}.$$

2d, By means of (159) we find, — For area  $b \times (h - \epsilon)$ ,

$$R_1 = \int_0^{h-\epsilon} y^2 dy = \frac{1}{8} \int_0^{h-\epsilon} (h-\epsilon)^3;$$

for area  $(b - b_i) \times \epsilon$ ,

$$R_2 = f(b - b_1) \int_0^b y^2 dy = \frac{1}{b} f(b - b_1) b^2;$$

for area  $b_i \times (h_i - \epsilon)$ ,

$$R_3 = fb_1 \int_0^{h_1-\epsilon} y^2 dy = \frac{1}{8} fb_1(h_1-\epsilon)^3.$$

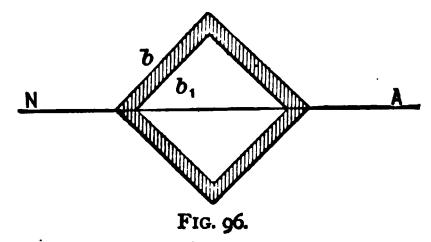
Whence the moment of resistance due to the net cross-section is

$$R = \frac{B}{3^{2}} [b(h-\epsilon)^{3} + (b-b_{1})\epsilon^{3} - b_{1}(h_{1}-\epsilon)^{3}], \quad (166)$$

where  $B = \epsilon f$  = unit strain at the extreme edge of the beam.

In a similar manner may all other beams be treated whose cross-sections are composed of rectangles having two sides parallel to the neutral axis.

57. Solid or Hollow Beam of Square Cross-Section and Diagonal Vertical.



Let b, Fig. 96, be a side of the beam's cross-section, and  $b_1$  a side of the square concavity whose centre coincides with the beam's centre. Then the diagonals are  $b\sqrt{2}$  and  $b_1\sqrt{2}$ ; and (159) becomes, since  $s = \frac{1}{2}b\sqrt{2} - y$ ,—

For solid beam,

$$R = 2f \int_{0}^{\frac{1}{2}\delta\sqrt{2}} 2(\frac{1}{2}\delta\sqrt{2} - y)y^{2}dy,$$

$$\therefore R = \frac{1}{12}fb^{4} = \frac{\sqrt{2}}{12}Bb^{3}, \qquad (167)$$

where  $B = \frac{1}{2} fb\sqrt{2} =$  intensity of stress at extreme upper or lower edge of the beam whose diagonal is vertical.

If in (160) we make h = b, then

$$R = \frac{1}{4\pi}fb^4 = \frac{1}{8}Bb^3,$$

where  $B = \frac{1}{2}fb =$  intensity of stress at upper or lower surface of a square beam whose side is vertical.

Hence, although the identity of the middle members of equations (160) and (167) shows that the total moment of resistance, R, is the same for a given solid square beam whether its side or its diagonal be vertical, yet the extreme fibres for these two positions of the beam are strained in the ratio of their distances from the neutral axes; that is, in the ratio of 1 to  $\sqrt{2}$ .

If, therefore, B expresses the ultimate strength of the material, when in equation (167) it is equal to  $\frac{1}{2}fb\sqrt{2}$ , we may evidently give to B the same extreme value in equation (160), and thus make the beam  $\sqrt{2}$  times stronger when its side is vertical than when its diagonal is vertical.

Again, for the vacant square whose side is  $b_1$ , since  $s = \frac{1}{2}b_1\sqrt{2} - y$ , we have

$$R = 2f \int_{0}^{\frac{1}{2}b_{1}\sqrt{2}} 2(\frac{1}{2}b_{1}\sqrt{2} - y)y^{2}dy,$$

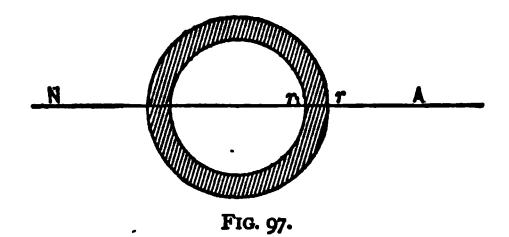
$$\therefore R = \frac{1}{12} f b_1^4.$$

And therefore, for a hollow square beam with diagonal vertical, the moment of resistance is

$$R = \frac{\sqrt{2}}{12} B \frac{b^4 - b_1^4}{b}, \tag{168}$$

where  $B = \frac{1}{2} f b \sqrt{2} = \text{unit strain at extreme edge of beam when the diagonal is vertical.}$ 

### 58. Solid or Hollow Beam of Circular Cross-Section.



Let r = radius of the outer circle, Fig. 97, and  $r_1 = \text{radius}$  of the inner circle. The equation of the outer circle is

$$r^2 = y^2 + s^2,$$

$$\therefore s = (r^2 - y^2)^{\frac{1}{2}};$$

and equation (159) becomes

$$R = 2f \int_{-r}^{+r} (r^2 - y^2)^{\frac{1}{2}} y^2 dy,$$

$$\therefore R = f \pi \frac{r^4}{4} = 0.7854 B r^3, \qquad (169)$$

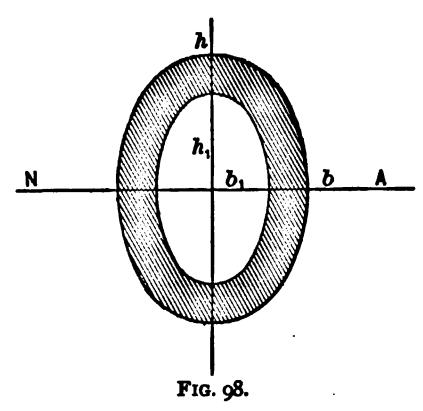
which is the moment of resistance for a solid beam of circular cross-section with the radius r, and where B = fr = the unit strain on the highest and lowest fibres.

If the beam is hollow, the inner and outer circles being concentric, we manifestly have

$$R = 0.7854B^{\frac{r^4}{r}} - \frac{r_i^4}{r}, \qquad (170)$$

where B = unit strain on highest and lowest fibres.

59. Beam of Elliptical Cross-Section, Solid or Hollow; Longer Axis vertical; Axes of Outer and Inner Ellipses coincident.



Let h, Fig. 98, be the length of the semi-transverse axis of the outer ellipse, and  $h_i$  that of the inner ellipse; b = lengthof semi-conjugate axis of outer ellipse, and  $b_i$  = the same for the inner ellipse. The equation of the outer ellipse is

$$\frac{z^2}{b^2}+\frac{y^2}{h^2}=1,$$

$$\therefore z = \frac{b}{h}(h^2 - y^2)^{\frac{1}{4}}.$$

Whence (159) becomes

$$R = \frac{2bf}{h} \int_{-h}^{+h} (h^2 - y^2)^{\frac{1}{2}} y^2 dy,$$

$$\therefore R = \frac{\pi}{4} fbh^3 = 0.7854Bbh^2, \tag{171}$$

which is the moment of resistance for the solid elliptical beam, where B = hf = unit strain on highest and lowest fibres.

Similarly, for the area of the inner ellipse,

$$R=\frac{\pi}{4}fb_{\rm i}h_{\rm i}^3;$$

and therefore, for the hollow elliptical beam, the moment of resistance is

$$R = 0.7854B \frac{bh^3 - b_1h_1^3}{h}, \qquad (172)$$

where B = hf = unit strain on highest and lowest fibres.

60. These illustrations may suffice for girders of continuous web.

We close this section with a table giving the limiting value of B, in pounds avoirdupois to the square inch, for the ordinary materials used in beams; that is, the values of B in this table are values which cannot be exceeded in the equations of this section, and represent the ultimate resistance of the material to cross-breaking.

It should, however, be borne in mind, that B may not represent the actual unit strain which the material is capable of resisting, either in tension or compression; but that it, in general, has some mean value between the ultimate resistance of material to crushing, and the ultimate resistance of the same material to tearing by direct pull.

Continuing the suppositions made in article 50, we may find the relation existing among these three ultimate unit strains in rectangular beams as follows:—

Let us take

h = depth of beam.

b = breadth of beam.

l = length of beam.

x =distance of the neutral surface from the compressed side of the beam.

C = ultimate resistance of the material to crushing by direct thrust, in pounds, per square inch.

T = ultimate resistance of the material to extension, in pounds, per square inch.

B = the unit strain which, at the instant of rupture, would be developed at the upper and lower surfaces of a beam having its neutral surface midway between these outer surfaces; that is, B = the modulus of rupture, also in pounds per square inch.

Then, using (159), we find for the compressed part of any cross-section, moment of internal forces,

$$R = \frac{1}{2}C\delta x^2; \qquad (173)$$

and for the extended part of the same cross-section,

$$R = \frac{1}{3} Tb(h - x)^2. \tag{174}$$

But, if the neutral axis bisected the given cross-section, we should have moment of internal forces on either side of this axis,

$$R = \frac{1}{3}B\delta(\frac{1}{2}h)^2.$$
 (175)

Now, since experimental researches show, that, for many materials used in construction, these three expressions are nearly equal to one another, we have approximately

$$\frac{1}{3}Cbx^2 = \frac{1}{3}Tb(h-x)^2 = \frac{1}{3}Bb(\frac{1}{2}h)^2; \qquad (176)$$

whence

$$x = \frac{1}{2}h\sqrt{\frac{B}{C}} = h\frac{\sqrt{T} - \frac{1}{2}\sqrt{B}}{\sqrt{T}} = h\frac{\sqrt{T}}{\sqrt{C} + \sqrt{T}},$$
 (177)

$$C = \frac{B}{\left(2 - \sqrt{\frac{B}{T}}\right)^2},\tag{178}$$

$$T = \frac{B}{\left(2 - \sqrt{\frac{B}{C}}\right)^2},\tag{179}$$

$$B = \frac{4C}{\left(1 + \sqrt{\frac{C}{T}}\right)^2} = \frac{4T}{\left(1 + \sqrt{\frac{T}{C}}\right)^2}.$$
 (180)

When, therefore, any two of the three quantities, C, T, and B, are given, the third may be found, and also the position of the neutral surface.

It is probable, that after the elastic limit of the material is passed, and rupture is about to take place, the expressions in (176) do not represent the actual moments, but are similar functions of C, T, and B, and are *proportional* to the *forces* then developed. For within the elastic limits the forces are

$$\frac{1}{2}C_{i}bx = \frac{1}{2}T_{i}b(h-x) = \frac{1}{2}B_{i}b(\frac{1}{2}h), \qquad (181)$$

 $C_n$ ,  $T_n$ , and  $B_n$  being the limited unit strains at the surfaces. But when the strain on the extreme fibres passes the elastic limit, and the fibres expand or contract more rapidly than the strain increases, then an increment is given to all the previous inner strains proportional to their distances from the neutral surface, which is equivalent to introducing a factor of the form kx into the expressions (181), whereby, at the instant of rupture, they become

$$\frac{1}{2}Cbkx^2 = \frac{1}{2}Tbk(h-x)^2 = \frac{1}{2}Bbk(\frac{1}{2}h)^2, \qquad (182)$$

which is identical with (176).

Experiments indicate, that in the case of cast-iron, owing to the superior hardness of the outer over the inner portions of the metal, an increment should be given to B in (180) equal to one-ninth of itself.

TABLE II.

	Authority.	Ultimate Resistance, in Lbs., per Square Inch, to			Modulus of
MATERIAL.		Com- pression, C.	Tension, T.	Cross- Breaking, B.	Elasticity, E.
Cast-Iron.					
Means of 9 irons	Stoney.	105945 H. 86284 H. - -	16720 H. 15298 H. - -	37695 H.F. 45696 C. 30240 C. 26880 C.	- 12000000 - - -
Circular tubes		112000 87429 127323	 16500 20500 34407	38304 C. 45965 C. 38250 45760 67035	- 17000000 11450754 15968254
Wrought-Iron.					
Bars, new Bars, previously strained Bars, new round Boiler tubes, welded Circular tubes, riveted Rolled I-beams, about T-iron, flange up about T-iron, flange down about Average Bars and bolts Bars and bolts Plates Plates, double-riveted Plates, single-riveted Hoops, best-best Wire Wire Wire Plate beams	Stoney.	    40320 36000 40000    	57555 K. 60000 70000 51000 35700 28600 64000 70000 100000 90000	51341 C. 74995 C. 30240 C. 70291 C. 43814 C. 61824 53760 51475 52567 C.	24000000 
Steel.  Bessemer, hammered  Bessemer, rolled	Stoney.	225568 F. -	83391 F. 71658 K.	128083 K. 115181 K.	31000000

TABLE II. — Continued.

Material.	Authority.	Ultimate Resistance, in Lbs., per Square Inch, to  Compression, Tension, Cross-Breaking, B.		Modulus of Elasticity, E.	
Steel (continued).  Crucible, hammered  Crucible, rolled  Cast, not hardened  Cast, low temper  Cast, mean temper	Stoney.	- 198944 Wd. 354544 Wd. 391985 Wd.	8 <sub>5</sub> 546 K. 68589 K. -	147840 K. 118272 K. - -	
Cast, high temper	Rankine.	372598 Wd. - - - -	100000 130000 80000	- - - 127344	 29000000 42000000 
Alder Ash Ash Beech Beech Beech Birch, American Birch, English Box Cedar, American white Cedar of Lebanon Chestnut, Spanish Chestnut	Stoney.  Rankine.  Stoney.  Rankine.  Stoney.  Rankine. Stoney.  Rankine. Stoney.  Kankine. Stoney.  Kankine.	6831 H. 9363 H. 9000 11500 9363 H 11663 H. 6402 H 10300 - 5863 H. 5860 5350	13900 Mu. 16700 Bv. 17000 B. 9360 11500 B. 17300 Mu. - 15000 Bv. 20000 B. 20000 Bv. 11400 Bv. 11400 Bv. 11400 Bv. 11500 Bv. 12100 Bv. 12100 Bv.	- 12156 B. 13000 10500 9336 B 12366 B.D. 11568 B. 14670 T 4596 D. 8958 D. 7400 - 10660 - 9372 B.	1600000 1350000 1645000 486000 1140000
Deal, red	« « Rankine. Stoney. «	6586 H. 7293 H. 10300 10331 H.	- 14000 14400 By, -	7850 - 4692 B.D. 11820 D.N.	- - 1020000 - - -

TABLE II. — Continued.

			<del></del>		70 Barriera
		Ultimate Resistance, in Lbs., per Square Inch, to			Modulus of
Material.	Authority.	Com- pression, C.	Tension, T.	Cross- Breaking, B.	Elasticity, E.
Wood (continued).					
Fir, spruce	Stoney.	6819 H.	_	8076 M.	
Fir, spruce, American black	«	- COLY 11.	_	6216 D.	_
Fir, Mar forest	ce	_	12000 B.	7392 B.	_
Fir, red pine	Rankine.	5375	12000	7392 D. 7100	1460000
Fir, red pine	44	5373 6200	14000	9540	1900000
Fir, yellow pine, American .	44	5400	-	-	-
Fir, spruce	44	3400	12400	9900	1400000
Fir, spruce	<b>44</b>	_	-	12300	1800000
Fir, larch	ee	5570	9000	5000	900000
Fir, larch	66	33/-	10000	10000	1360000
Hemlock	Stoney.	_	•	6852 D.	-
Hickory, American	"		•	12774 D.N.	-
Hickory, bitter-nut · · ·	44	-	-	8790 D.	-
Larch	<b>66</b>	5568 H.	10220 Ro.	8010 B.D.	
Larch	44	_	8900 Bv.	_	_
Larch, American	46	-	-	5466 D.	-
Lignum-vitæ	ec	_	11800 Bv.	12078 N.	-
Lignum-vitæ	Rankine.	9900	11800	12000	-
Locust	68	-	16000	11300	-
Locust	Stoney.	-	20100 Mu.	20580 B.	-
Locust	Haswell.	9113	-	-	-
Mahogany	Rankine.	-	8000	7600	1255000
Mahogany	**	8200	21800	11500	-
Mahogany	Stoney.	8198 H.	8000 B.	-	-
Mahogany	44	-	16500 Bv.	10314 M.N.	-
Maple	66	-	17400 Bv.	10164 D.	-
Maple	Rankine.	-	10600	-	-
Maple	Haswell.	8150	-		-
Oak, European	Rankine.	7700	10000	8700	1200000
Oak, European	44	10000	19800	13600	1750000
Oak, American red	"	6000	10250	10600	2150000
Oak, English	Stoney.	10058 H.	10000 B.	10164 B.D.	-
Oak, English	"	-	19800 Bv.	-	-
Oak, French	"	0 77	13950 Ro.	8898 M.	-
Oak, Quebec	" "	5982 H,	- I	- TO 33	-
Oak, American red		-	•	10122 D.N.	_
Oak, American white	**		-	10458 B.D	
Pine, American red		7518 H.	-	9162 B.D.	- 1
Pine, American pitch Pine, American white	**	6790 H.	7650 Mu.	10362 B.D.	_
ine, militali wille	ŀ	- !	_	7374 D.N.	

TABLE II. — Continued.

		Ultimate Resistance, in Lbs., per Square Inch, to			Modulus of
MATERIAL.	Authority.	Com- pression, C.	Tension, T.	Cross- Breaking, B.	Elasticity, E.
Wood (continued).					
Pine, American yellow	Stoney.	5445 H.	<b>-</b>	7110 B.D.	_
Pine, Norway	"	-	14300 Bv.	-	_
Pine, Norway	ee	-	7287 Bv.	-	_
Sycamore	Rankine.	-	13000	9600	1040000
Sycamore	Stoney.	7082 H.	13000 Bv.	<b>–</b>	-
Teak	44	12101 H.	15000 B.	12648 B.M.	_
Teak, Indian	Rankine.	12000	15000	12000	2400000
Teak, Indian	66	-	_	19000	-
Teak, African	44	-	-	14980	-
Walnut	Stoney.	7227 H.	8130 Mu.	_	-
Walnut	66	- [	78∞ Bv.	-	-
Willow	<b>66</b>	6128 H.	14000 Bv.	••	-
Willow	Rankine.	- 1	-	6600	-
Stone.	Carre	377		317	
Granite	Stoney.	3173 Wi.	-	456 Wi.	-
Granite	Dankina	13440 Wi.	-	2442 Wi.	~
Granite	Rankine.	5500	_	-	-
Granite	44	11000	-	-	
Limestone	44	4000	_	_	_
Limestone	Stoney.	4500 3050 F.	_	1698 Wi.	_
Limestone	stoney.	3050 F. 18043 Wi.	_	2484 Wi.	_
Limestone	Haswell.	20043 111.	670	-4~4 ** T*	_
Limestone	11 diswell.		2800		_
Marble	Stoney.	3216 Re.	551 H.	-	-
Marble	"	20160 Wi.	722 Bu.	-	_
Marble	Rankine.	5500		_	_
Marble, white	-	-	_	1252 H.	-
Marble, black	_	- 1	<b>-</b>	2697 H.	-
Marble, black	Moseley.	-	- !	2664	-
Sandstone	Stoney.	2185	2054 Bu.	2010 Re.	-
Sandstone	44	7884	1261 Bu.	5142 Re.	-
Sandstone	Rankine.	5500	-	2360	-
Sandstone	46	2200	-	1100	-
Slate	•	-	9600	5000	13000000
Slate	66	-	12800	-	16000000
Slate	Stoney.	≖7344	-	7370	-

TABLE II. - Concluded.

Material.	Authority.	Ultimate Resistance, in Lbs., per Square Inch, to			Modulus of
		Com- pression, C.	Tension, T.	Cross- Breaking, B.	Elasticity, E.
Bricks, etc.					
Pale red	Stoney.	562 Re.		-	_
Red	"	808 Re.	-	-	_
Fire	"	1717 Re.	-	-	_
Gault clay	"	2240 Gr.	-	-	1 -
Ordinary	Rankine.	_	280	-	_
Ordinary	**	-	300	-	-
Lime mortar, average	Stoney.	618 Ro.	51	-	-
Portland cement	44	5984 Gr.	358 Gr.	-	_
Plaster of Paris	64	-	71 Ro.	-	_
ROMAN CEMENT: -					Į
2 years	44	- 1	546 Gr.	-	-
3 years	46	-	604 Gr.	-	-
4 years	**	-	632 Gr.	-	-
5 years	44	-	627 Gr.	-	-
6 years	46	-	666 Gr.	-	-
7 years	46	-	709 Gr.	-	-

The value of B, the modulus of rupture in Table II., is that due to a rectangular cross-section, unless otherwise specified.

The works from which this Table is made up are the following well-known authorities:—

- 1st, "A Manual of Civil Engineering," by William John Macquorn Rankine.
- 2d, "The Theory of Strains in Girders and Similar Structures," by Bindon B. Stoney.
- 3d, "The Mechanical Principles of Engineering and Architecture," by Henry Moseley.
- 4th, "Engineers' and Mechanics' Pocket-Book," by Charles H. Haswell.
- 5th, "Report on the Progress of Work, etc., of the Cincinnati Southern Railway," by Thomas D. Lovett.

6th, "Report on Tests of Salisbury Cast-Iron," in "Railroad Gazette" of Nov. 30, 1877, by Robert H. Thurston.

Names of the experimenters cited are thus abbreviated: viz., H., Hodgkinson; F., Fairbairn; Bv., Bevan; Bu., Buchanan; B., Barlow; D., Denison; N., Nelson; M., Moore; K., Kirkaldy; Ro., Rondelet; Re., Rennie; C., Clark; Gr., Grant; Wi., Wilkinson; Wd., Wade; Mu., Musschenbroek; T., Trickett.

### SECTION 2.

Moment of Inertia and Radius of Gyration of a Given Cross-Section.

61. In equation (159) the factor

$$\iint y^2 \, dy \, dz = I \tag{183}$$

is called the *moment of inertia* of the surface of the cross-section, relatively to the axis of z, the factor being analogous to the real *moment of inertia* of a material plate whose thickness is unity.

The moment of inertia divided by the area of the section gives the *square* of the *radius of gyration*, which we will call  $r^2$ .

We then have, if S is that area,

Square of radius of gyration = 
$$r^2 = \frac{I}{S} = \frac{\iint y^2 \, dy \, dz}{\iint dy \, dz}$$
. (184)

62. From the moments of resistance already found, equations (160) to (171), and from similar applications of (183), we derive values of I and of  $r^2$  as given below in Table III., where the axes of gyration are assumed to pass through the centre of gravity of the cross-section.

TABLE III.

FORM OF CROSS-SECTION.		Moment of Inertia of Section, I.	Square of Radius of Gyration,
1. Rectangle (Fig. 90).  About least axis b  About greater axis h		1√26ħ³ 1√26³ħ	- 1√2 Ų. - 1√2 Ų.
2. Square. About b or h		12h4	<del>1</del> 12/62.
3. Hollow Rectangle (Fig. 91).			
About least axis b	Max.	$\frac{1}{12}(bh^3-b_1h_1^3)$	$\frac{bh^3 - b_1h_1^3}{12(bh - b_1h_1)}.$
About greater axis &	Min.	$\frac{1}{12}(b^3h-b_1^3h_1)$	
4. Hollow Square. About b or h		$\frac{1}{12}(h^4-h_1^4)$	$\frac{1}{12}(h^2+h_1^2).$
5. I-Section (Fig. 91).			10 4
About vertical axis h		12b2A	$\frac{b^2A}{12(A+B)}.$
About horizontal axis b		$\frac{1}{12}(bh^3-b_1h_1^3)$	
A = area of flanges. B = area of web.			12(11 + 2)
6. Plates and Channels (Fig. 92).	_		
About axis b, normal to plates.	{	$ \begin{array}{rcl} & \frac{1}{12}(b - b_2)h^3 \\ & + \frac{1}{12}b_2(h - 2h_2)^3 \\ & - \frac{1}{12}b_1h_1^3 \end{array} $	I÷S.
About axis $b$ , parallel to plates $\{Fig. 94\}$	<b>\</b>	$\frac{1}{12}(bh^3 - b_1h_1^3 - b_2h_2^3)$	I÷S.
7. Plates and I-Beams (Fig. 93).			
About axis $\delta$ , parallel to plates.	<b>\</b>	$\frac{\frac{1}{12}(bh^3 - b_1h_1^3)}{-b_2h_2^3}$	I÷S.
About axis h, normal to plates.		$ \frac{1}{6}(h - h_2)b^3  + \frac{1}{12}(b - \frac{1}{2}b_2)^3 h_2  - \frac{1}{12}(b_1 + \frac{1}{2}b_2)^3 h_2  - \frac{1}{6}(h_1 - h_2)b_1^3 $	I ÷ S.

TABLE III. - Concluded.

Form of Cross-Section.		Moment of Inertia of Section,	Square of Radius of Gyration,
8. T-Section, erect (Fig. 95).  About axis b, parallel to flange $\varepsilon = \frac{bh^2 - b_1h_1^2}{2(bh - b_1h_1)} = \text{height of }$ neutral axis.	l	$ \frac{\frac{1}{3}b(h-\epsilon)^3}{+\frac{1}{3}(b-b_1)\epsilon^3}  -\frac{1}{3}b_1(h_1-\epsilon)^3 $ $ \frac{1}{3}(h-h_1)b^3 $	I÷S.
About axis $h$ , normal to flange.	{	$\frac{1}{12}(h - h_1)b^3 + \frac{1}{12}h_1(b - b_1)^3$	$I \div S$ .
9. Angle Iron; equal ribs b, thickness = t	Min.	$\frac{\frac{1}{12}tb^{3}}{b^{2}h^{2}(b+h)t}$ $\frac{b^{2}h^{2}(b+h)t}{12(b^{2}+h^{2})}$	$\frac{\frac{1}{14}b^2}{b^2h^2}$ $\frac{b^2h^2}{12(b^2+h^2)}$
10. Channel Iron; $h = \text{depth of }$ flanges $+\frac{1}{2}$ thickness of web, $A = \text{area of flanges}, B = $ area of web	Min.	$h^2\left(\frac{A}{12}+\frac{AB}{4S}\right)$	
11. Star Iron, or cross of equal arms h	Min.	14Sh2	$\frac{1}{24}h^2$ .
12. Ellipse (Fig. 98).  About minor axis 2b  About major axis 2h		₹πόλ³ ₹πό³λ	$\frac{1}{4}h^2$ . $\frac{1}{4}b^2$ .
13. Circle; radius h (Fig. 97)		$\frac{1}{4}\pi h^4$	¼h².
14. Hollow Ellipse (Fig. 98).  About minor axis 26	Max.	$\frac{\pi}{4}(bh^{3} - b_{1}h_{1}^{3})$ $\frac{\pi}{4}(b^{3}h - b_{1}^{3}h_{1})$	$\frac{1}{4} \frac{\partial h^3 - b_1 h_1^3}{\partial h - b_1 h_1}.$
About major axis 2h	Min.	$\frac{\pi}{4}(b^3h-b_1^3h_1)$	$\left  \frac{1}{4} \frac{b^3h - b_1^3h_1}{bh - b_1h_1} \right $
15. Hollow Circle (Fig. 97).			
Radii, h, h <sub>1</sub>		$\frac{\pi}{4}(\hbar^4-\hbar_1^4)$	$\frac{1}{4}(h^2+h_1^2).$

## CHAPTER VI.

DEFLECTION, END MOMENTS, AND POINTS OF CONTRARY FLEXURE FOUND. — CAMBER.

## SECTION I.

Deflection of the Semi-Beam having a Uniform Cross-Section.

63. Equation of the Elastic Curve as applied to a Beam or Pillar. — Let N, Fig. 89, be the origin, and x and y the current co-ordinates, of the neutral line NTS of any beam or column under a given load; TU = a unit of the length of the beam; VT = VU = a the radius of curvature at any point a

$$\rho = -\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = -\frac{1}{\frac{d^2y}{dx^2}}$$
 (185)

when the deflection of the beam or pillar is small compared with its length. (See Differential Calculus.)  $PP_1 = \text{increment of unit on extended side due to flexure; } RR_1 = \text{decrement of unit on compressed side due to flexure; } \alpha = \text{the angle included between the tangent to the curve at any point and the axis of <math>x$ , so that  $\tan \alpha = \frac{dy}{dx}$ .

Suppose the unit strain required to extend a unit of length by the space  $PP_1$  to be  $T_1$ , and that required to compress a

unit of length by the space  $RR_i$  to be  $C_i$ , and that required to extend or compress a unit of length by its own length, TU, to be E = the modulus of transverse elasticity; and put  $C_i + T_i = 2B_i$  = total unit strain at surfaces.

We then have, if h is the depth of the beam, and if the displacing forces E and 2B, are proportional to the displacements they cause,

$$\frac{P_{i}P}{PU} = \frac{R_{i}R}{RU} = \frac{TU}{UV},$$

$$\therefore \frac{2B_{i}}{h} = \frac{E}{\rho} = -E\frac{d^{2}y}{dx^{2}}.$$
(186)

Multiplying (186) by I = moment of inertia of cross-section, we find

$$\frac{2B_x I}{h} = -EI \frac{d^2 y}{dx^2} = M_x = R, \qquad (187)$$

which is the moment of resistance of the cross-section, since  $B_1 \div \frac{1}{2}h = f$ , and  $I = \iint y^2 dy dz$  of equation (159).

If, therefore, we put  $-EI\frac{d^2y}{dx^2}$  equal to the moment at the section due the external forces acting on a beam or pillar of uniform cross-section, and perform two successive integrations, we shall have an equation in which y is the deflection of the neutral line at the distance x from the origin of co-ordinates.

64. Deflection of the Semi-Beam under One Weight.— Let L, Fig. 8, the point where the neutral line of the semi-beam meets the wall, be the origin of co-ordinates, and call x positive to the left, and y positive downward, in accordance with the notation in article 14.

Take semi-beam of length l, with concentrated load W at distance a' from fixed end.

From equation (18),

$$M_x = -W(a'-x) = -EI\frac{d^2y}{dx^2}$$

by (187),

$$\therefore EI\frac{d^2y}{dx^2}=W(a'-x).$$

Integrating, with the condition that  $\frac{dy}{dx} = 0$  when x = 0, we have

$$EI\frac{dy}{dx} = W\left(a'x - \frac{x^2}{2}\right).$$

Integrating again, with the condition that y = 0 when x = 0.

$$\therefore EIy = W\left(a'\frac{x^2}{2} - \frac{x^3}{6}\right).$$

Deflection at any point,  $\dot{x}$ , is

$$y = \frac{W}{EI}(\frac{1}{2}a'x^2 - \frac{1}{6}x^3). \tag{188}$$

And when x = a' = l, we have

Maximum deflection 
$$D = \frac{W7^3}{3EI}$$
, (189)

where the proper values of E and I are to be taken from Tables II. and III., according to the material used, and the form of cross-section.

65. Semi-Girder, Length I, Uniform Load w per Unit.— From equations (23) and (187),

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}w(l-x)^2.$$

When 
$$\frac{dy}{dx} = 0$$
,  $x = 0$ ;

$$\therefore EI\frac{dy}{dx} = \frac{1}{2}wPx - \frac{1}{2}xx^2 + \frac{1}{6}wx^3.$$

When y = 0, x = 0;

$$\therefore EIy = \frac{1}{4}wl^2x^2 - \frac{1}{6}wlx^3 + \frac{1}{24}wx^4.$$

Deflection anywhere,

$$y = \frac{w}{24EI}(6l^2x^2 - 4lx^3 + x^4). \tag{190}$$

Maximum deflection,

$$D = \frac{w^{n}}{8EI} \text{ when } x = l. \tag{191}$$

66. Semi-Girder, Partial Uniform Load, w', on each Unit of Length, b, at Distance a from Fixed End. Fig. 8.— When x = a, or x < a, equations (29) and (187) give

$$EI\frac{d^2y}{dx^2}=w'b(\frac{1}{2}b+a-x).$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0,$$

$$\therefore EI\frac{dy}{dx} = w'b\left\{ (\frac{1}{2}b + a)x - \frac{x^2}{2} \right\}.$$

y = 0 when x = 0,

:. 
$$EIy = w'b \left\{ (\frac{1}{2}b + a)\frac{x^2}{2} - \frac{x^3}{6} \right\}.$$

Deflection x not > a,

$$y = \frac{w'b}{EI} \left\{ (\frac{1}{2}b + a)\frac{x^2}{2} - \frac{x^3}{6} \right\}.$$
 (192)

Again,  $\frac{dy}{dx} = \tan \alpha$  when x = a,

$$\therefore EI\left(\frac{dy}{dx} - \tan \alpha\right) = w'b\left\{\left(\frac{1}{2}b + a\right)(x - a) - \frac{x^2 - a^2}{2}\right\}.$$

y = 0 when x = 0,

$$\therefore EI(y-x\tan\alpha)=w'b\Big\{(\frac{1}{2}b+a)\Big(\frac{x^2}{2}-ax\Big)-\Big(\frac{x^3}{6}-\frac{a^2x}{2}\Big)\Big\}...$$

Let  $y = y_1$  when  $x = a_1$ 

$$\therefore EIa \tan \alpha - EIy_1 = \frac{1}{4}w'a^3b^2 + \frac{1}{8}w'a^3b.$$

From (192),

$$EIy_1 = \frac{1}{4}w'a^2b^2 + \frac{1}{8}w'a^3b,$$

$$\therefore EI \tan a = \frac{1}{2}w'ab(b+a).$$

Also, when x is not < a nor > (a + b), we have, from (26) and (187),

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}w'(a+b-x)^2 = \frac{1}{2}w'(a+b)^2 - w'(a+b)x + \frac{1}{2}w'x^2.$$

$$\frac{dy}{dx} = \tan \alpha \text{ when } x = a,$$

$$EI\left(\frac{dy}{dx} - \tan \alpha\right) = w'\left\{(a+b)^{2}\frac{x-a}{2} - (a+b)\frac{x^{2}-a^{2}}{2} + \frac{x^{3}-a^{3}}{6}\right\}.$$

$$y = y_1 \text{ when } x = a,$$

$$\therefore EI(y - y_1) - (x - a) \tan \alpha EI$$

$$= \frac{w'}{6} \left\{ 3(a + b)^2 \left[ \frac{x^2 - a^2}{2} - a(x - a) \right] \right\}$$

$$-3(a+b)\left[\frac{x^3-a^3}{3}-a^2(x-a)\right] + \frac{x^4-a^4}{4}-a^3(x-a)\right\}.$$

Whence, after eliminating  $y_1$  and tan  $\alpha$ , we find the deflection

$$y = \frac{w'}{24EI} [x^4 - a^4 - 4(a+b)(x^3 - a^3) + 6(a+b)^2(x^2 - a^2) - 4a^3(x-a) + 6a^2b^2 + 8a^3b].$$
 (193)

And if x = a + b, we have

Maximum deflection 
$$D = \frac{w'}{24EI}[3(a+b)^4 - 3a^4 - 4a^3b].$$
 (194)

67. If it be required to find the total deflection of a semibeam at its free extremity when it supports partial uniform or concentrated loads not reaching that extremity, we may proceed as follows:—

Find the deflection at the free end due the beam's own weight, lw; then find the deflection, D, and the inclination,  $\alpha$ , of the beam at any point bearing a concentrated load, W, or at the outer end of any partial uniform load, bw'; using  $\tan \alpha$ , compute the end deflection due W or bw' by the formula

$$\begin{cases} D_1 = D_{w'} + (l - a - b) \tan \alpha \text{ for } bw', \\ D_2 = D_W + (l - a') \tan \alpha \text{ for } W. \end{cases}$$

These deflections added to that due the beam's own weight, will give the total deflection at the free end of the semi-beam.

Example. — Suppose the semi-beam, Fig. 8, to be of oak, weighing 54 pounds to the cubic foot, and to have a rectangular cross-section 1 foot deep and  $\frac{1}{2}$  foot wide, so that weight of beam per foot of length = w = 27 pounds; length = 15 feet, b = 4 feet, loaded with w' = 100 pounds per foot, beginning a = 5 feet from the fixed end of beam; W = 100 pounds, placed a' = 11 feet from fixed end; E = 2,000,000 pounds per square inch = 288,000,000 pounds per square foot.

From Table III.,  $I = \frac{1}{12}bh^2 = \frac{1}{12} \times \frac{1}{2} \times 1^2 = \frac{1}{24}$ ,

$$EI = 12000000.$$

Deflection due beam's own weight is, by (191),

$$D_w = \frac{27 \times 15^4 \times 12}{8 \times 12000000} = 0.17086$$
 inch.

From (194),

$$D_{w'} = \frac{100[3(5+4)^4 - 3 \times 5^4 - 4 \times 5^3 \times 4]}{24 \times 120000000} \times 12 = 0.06587 \text{ inch.}$$

Differentiating (193), we have

$$\frac{dy}{dx} = \tan \alpha_1$$

$$= \frac{w'}{24EI} [4x^3 - 12(a+b)x^3 + 12(a+b)^2x - 4a^3]$$

$$= \frac{100 \times 2416}{24 \times 12000000}$$

when x = a + b = 9;

$$\therefore (l-a-b)\tan \alpha_1 = \frac{6 \times 100 \times 2416 \times 12}{24 \times 12000000} = 0.0604 \text{ inch,}$$

:.  $D_1 = 0.06587 + 0.06040 = 0.12627$  inch at end due bw.

From (189),

$$D_W = \frac{100 \times 15^3 \times 12}{3 \times 12000000} = 0.1125 \text{ inch.}$$

Differentiating (188), we find

$$\frac{dy}{dx} = \tan \alpha_2 = \frac{W}{EI} \left( a'x - \frac{x^2}{2} \right) = \frac{100 \times \frac{1}{2} \times 121}{120000000}$$

when x = a' = 11;

$$\therefore (l-a')\tan a_2 = \frac{4 \times 100 \times \frac{1}{2} \times 121 \times 12}{12000000} = 0.0242 \text{ inch,}$$

$$D_2 = 0.1125 + 0.0242 = 0.1367$$
 inch at end due  $W$ .

Therefore total deflection at free end is

$$D = D_w + D_1 + D_2 = 0.17086 + 0.12627 + 0.1367 = 0.43383$$
 inch.

68. If we have any number, r, equal weights, W, placed at equal intervals,  $\frac{l}{n}$ , along the semi-girder, the first weight being a full interval from the fixed end, and the  $n^{\text{th}}$  or last weight being at the free end when the beam is fully loaded, then the total deflection at the free end due to the first r equal weights may be found as follows:—

In equation (188) put 
$$x = a' = \frac{lr}{n}$$
; then
$$y = \frac{W}{3EI} \left(\frac{l}{n}\right)^3 r^3, \tag{195}$$

which is the deflection at the  $r^{th}$  weight due to that weight alone.

But the deflection at the free end due the  $r^{th}$  weight is greater than the deflection at the  $r^{th}$  weight by the product of the tangent of the slope  $\alpha$  of the beam at and beyond the  $r^{th}$  weight, multiplied by the horizontal distance between this weight and the free end; and this horizontal distance is practically equal to  $\left(l - \frac{lr}{n}\right)$  = the part of the beam's length beyond the  $r^{th}$  weight.

By differentiating (188), we find

$$\frac{dy}{dx} = \tan \alpha = \frac{W}{EI} \times \frac{1}{2} \left(\frac{hr}{n}\right)^{2}$$
if  $x = \alpha' = \frac{lr}{n}$ , and
$$\left(l - \frac{lr}{n}\right) \tan \alpha = \frac{W}{EI} \left(\frac{l}{n}\right)^{3} \left(\frac{1}{2}nr^{2} - \frac{1}{2}r^{3}\right).$$

Add value of y in (195) =  $\frac{W}{EI} \left(\frac{l}{n}\right)^3 \frac{r^3}{3}$ , and we have deflection at free end due the  $r^{\text{th}}$  weight,

$$D_r = \frac{Wl^3}{EI} \left( \frac{r^2}{2n^2} - \frac{r^3}{6n^3} \right). \tag{196}$$

Hence the end deflection due the first r weights is

$$\hat{D}_{2r} = \frac{W7^3}{EI} \left\{ \frac{1}{2n^2} (1^2 + 2^2 + 3^2 + 4^2 \dots + r^2) - \frac{1}{6n^3} (1^3 + 2^3 + 3^3 + 4^3 \dots + r^3) \right\} 
= \frac{W}{24EI} \left( \frac{l}{n} \right)^3 \left[ 2nr(r+1)(2r+1) - r^2(r+1)^2 \right]. \quad (197)$$

When r = n, (197) becomes

$$D = \frac{W7^3}{24EI} \frac{(n+1)(3n+1)}{n}, \qquad (198)$$

which is the deflection at the free end due n equal weights at equal intervals,  $\frac{l}{n}$ .

69. If the weight at the free end is but  $\frac{1}{2}W$ , while every other weight is W, as is generally the case with a uniform load of panel weights, we must diminish the deflection last found by the deflection due  $\frac{1}{2}W$  at the free end, which, according to (189), is

$$D = \frac{\frac{1}{2}Wl^3}{3EI}.$$

This quantity taken from the value of D in (198) leaves

$$D = \frac{W7^3}{24EI} \left\{ \frac{(n+1)(3n+1)}{n} - 4 \right\}, \quad (199)$$

which is the deflection at the free end of a semi-girder of uniform cross-section when the load is distributed in equal panel weights; there being but the half panel weight at the free end.

70. To find the Deflection at the  $r^{\text{th}}$  Interval due to all the n Equally Distributed Equal Weights, W.

For the first r weights, (198) applies if we put r for n, and l, for l; l, being the distance of the r<sup>th</sup> weight from the fixed end. That is,

$$D = \frac{Wl_1^3}{24EI} \frac{(r+1)(3r+1)}{r}.$$
 (200).

For the (n-r) weights beyond the  $r^{th}$ , we have, from (188), since x now becomes  $l_1$ ,

$$y = \frac{W}{EI}(\frac{1}{2}a'l_i^2 - \frac{1}{6}l_i^3), \qquad (201)$$

the deflection at the  $r^{th}$  weight due to any one weight at the distance a' from the fixed end,

Therefore, by summing (201), we find

$$D_{n-r} = \frac{W}{EI} \left\{ \frac{1}{2} l_1^2 \left( \frac{l}{n} [(r+1) + (r+2) + (r+3) \dots (r+n-r)] \right) - \frac{(n-r) l_1^3}{6} \right\}$$

$$= \frac{W}{EI} \left\{ \frac{1}{2} \frac{l^2 r^2}{n^2} \left[ \frac{l}{n} \left( r(n-r) + \frac{(n-r)(n-r+1)}{2} \right) \right] - \frac{(n-r) l^3 r^3}{6n^3} \right\}$$

$$= \frac{W}{EI} \left\{ \frac{1}{3} \frac{l^3 r^3}{n^3} (n-r) + \frac{1}{4} \frac{l^3 r^2}{n^3} (n-r) (n-r+1) \right\}, \qquad (202)$$

the deflection at the  $r^{th}$  weight due all the weights beyond. Adding (200) and (202), there results

$$D = \frac{Wl^3r^2}{24EIn^3}(6n^2 + r^2 - 4rn - 2r + 6n + 1), \quad (203)$$

which is the deflection at the  $r^{th}$  weight due to all the n given weights, W.

71. If the weight at the free end is the  $m^{th}$  part of W, instead of W, the deflection due  $\frac{W}{m}$  at the  $r^{th}$  point of division is, by putting a' = l, and  $l_1 = \frac{lr}{n}$ , in (201),

$$y = \frac{W}{mEI} \left( \frac{l^3 r^2}{2n^2} - \frac{l^3 r^3}{6n^3} \right) = \frac{Wl^3 r^2}{24EIn^3} \left( \frac{12n - 4r}{m} \right).$$

Subtracting this value of y from the deflection in (203), we have, finally,

$$D = \frac{W7^3r^2}{24EIn^3} \left(6n^2 + r^2 - 4rn - 2r + 6n + 1 - \frac{12n - 4r}{m}\right)$$
$$= \frac{W7^3r^2}{24EIn^3} \left(6n^2 - 4nr + r^2 + 1\right) \quad (204)$$

if m = 2; so that (204) is the deflection at the  $r^{th}$  panel point due to a full load of equal panel weights.

EXAMPLE. — Take the oak semi-beam of the example in article 67, and suppose it loaded with n = 5 equal weights, of 100 pounds each, at intervals of 3 feet. EI = 12,000,000.

Then the deflection at the free end due the 5 weights is, by equation (198),

$$D = \frac{100 \times 15^{3}}{24 \times 12000000} \times \frac{(5+1)(3 \times 5+1)}{5}$$
= 0.0225 foot = 0.27 inch;

to which add 0.17086, the deflection due the beam's own weight, for the total deflection at the free end = 0.44086 inch.

If the fifth or end weight is but  $\frac{1}{2} \times 100 = 50$  pounds, then, by (199),

$$D = \frac{100 \times 15^{3}}{24 \times 12000000} \left\{ \frac{(5+1)(3 \times 5+1)}{5} - 4 \right\}$$
= 0.0178125 foot = 0.21375 inch;

and the total deflection = 0.21375 + 0.17086 = 0.38461 inch.

To find the deflection at the third loaded point due the 5 equal weights in their positions, we use (203), taking r=3; thus,

$$D = \frac{100 \times 15^{3} \times 3^{2}}{24 \times 12000000 \times 5^{3}} (6 \times 5^{2} + 3^{2} - 4 \times 3 \times 5 - 2 \times 3 + 6 \times 5 + 1)$$

$$= 0.0104625 \text{ foot} = 0.12555 \text{ inch.}$$

And the deflection at this third interval, due the beam's own weight of 27 pounds per linear foot, is, by equation (190), putting x = 9, w = 27,

$$y = \frac{27}{24 \times 12000000} (6 \times 15^2 \times 9^2 - 4 \times 15 \times 9^3 + 9^4)$$
  
= 0.006766 foot = 0.081192 inch.

Therefore the total deflection at the third interval is

$$0.125550 + 0.081192 = 0.206742$$
 inch.

## SECTION 2.

Deslection of a Beam of Uniform Cross-Section, supported at its Free Unfixed Ends.

72. Deflection due the Beam's own Weight, supposed to be Uniform. — For the cases in this section we employ the same notation as that given in article 17, Fig. 9, excepting that we take the origin of co-ordinates at  $O_1$ , a point in the neutral surface, instead of using O as before, in order that y may be the deflection of the neutral line, as it is in the expression for the moment of the internal forces,  $R = -EI\frac{d^2y}{dx^2} = M$ , of article 63. We now have x positive to the right, and y positive downwards.

From equations (49) and (187),

$$-\frac{1}{2}w(l-x)x=EI\frac{d^2y}{dx^2}.$$

Integrating, with the condition that  $\frac{dy}{dx} = 0$  when  $x = \frac{1}{2}l$ ,

$$EI\frac{dy}{dx} = \frac{1}{2}w\left\{\frac{x^3 - (\frac{1}{2}l)^3}{3} - \frac{1}{2}l[x^2 - (\frac{1}{2}l)^2]\right\}.$$

Integrating again, with the condition that y = 0 when x = 0, we have, after reducing,

$$y = \frac{w}{24EI}(x^4 - 2lx^3 + l^3x), \qquad (205)$$

which is the deflection of the uniformly loaded beam at any point, w being the load per unit of the beam's length, L

If in (205) we put  $x = \frac{1}{2}l$ , we have

$$D = \frac{5wl^4}{384EI},$$
 (206)

the deflection at the centre due a continuous uniform load, lw.

Example. — Beam of oak, 54 pounds per cubic foot. Length 15 feet = l. Rectangular cross-section, depth 1 foot = h; breadth  $\frac{1}{2}$  foot = b. E = 2,000,000 pounds per inch.

.. 
$$E = 288000000$$
 pounds per square foot,  
 $I = \frac{1}{12}bh^3 = \frac{1}{24}$ ,  $EI = 12000000$ ;

all dimensions in feet.

Weight per foot of length =  $1 \times \frac{1}{2} \times 1 \times 54 = 27$  pounds. Deflection due beam's own weight at a point 5 feet from either end, by (205), is

$$y = \frac{27}{24 \times 12000000} (5^4 - 2 \times 15 \times 5^3 + 15^3 \times 5)$$
= 0.001289 foot.

From (206), the central deflection is

$$D = \frac{5 \times 27 \times 15^4}{384 \times 12000000} = 0.001483 \text{ foot.}$$

73. Deflection due a Concentrated Load, W, placed at the Horizontal Distance  $\alpha'$  from the Origin or End of the Beam. — When  $x < \alpha'$ , equations (40) and (187) apply; that is,

$$EI\frac{d^2y}{dx^2} = -W\frac{l-a'}{l}x.$$

Let  $\alpha$  be the angle of inclination, or slope, of the beam at the point of application of W. Then, integrating,  $\frac{dy}{dx} = \tan \alpha$  when x = a',

$$\therefore EI\left(\frac{dy}{dx} - \tan\alpha\right) = W\frac{(a'-l)}{2l}(x^2 - a'^2).$$

Again, y = 0 when x = 0,

$$\therefore EI(y-x\tan\alpha)=\frac{W(a'-l)}{2l}\left(\frac{x^3}{3}-a'^2x\right). \quad (207)$$

But when x > a', use equations (43) and (187), giving

$$EI\frac{d^2y}{dx^2} = \frac{Wa'}{I}(x-I).$$

Integrating between the limits  $\tan \alpha$  and  $\frac{dy}{dx}$ , and a' and x,

$$EI\left(\frac{dy}{dx}-\tan\alpha\right)=\frac{Wa'}{l}\left\{\frac{x^2-a^2}{2}-l(x-a')\right\}.$$

Integrating again between the limits 0 and y, and l and x, there results

$$EI[y - (x - l) \tan \alpha] = \frac{Wa'}{l} \left\{ \frac{x^3 - l^3}{6} - \frac{l(x^2 - a'^2)}{2} + (a'l - \frac{1}{2}a'^2)(x - l) \right\}. \quad (208)$$

By putting x = a' and  $y = y_1$  in equations (207) and (208), we find

$$\tan \alpha = \frac{Wa'}{EIl} \left( \frac{l^2}{3} - a'l + \frac{2}{3}a'^2 \right).$$

Putting this value of  $\tan \alpha$  in (207), we get, after reducing,

$$y = \frac{W(l-a')}{6EIl}[(2l-a')a'x - x^3], \qquad (209)$$

which is the deflection at any point between the origin and the weight W.

If x = a', we have the deflection at the loaded point,

$$D = \frac{Wa'^2}{3EIl}(l-a')^2.$$
 (210)

And if  $x = a' = \frac{1}{2}l$ ,

$$\tan \alpha = 0$$
;

and

Deflection at centre = 
$$D = \frac{W7^3}{48EI}$$
, (211)

which is a maximum, since W is now at the centre.

Comparing (211) and (206), where wl = W = entire load on the beam, we see that the deflection at the centre due the load, lw, uniformly distributed continuously, is five-eighths of the deflection at the centre due the same amount of load concentrated at that point.

By putting l-a' for a', and l-x for x, in (209), or by substituting the value of  $\tan \alpha = \frac{Wa'}{EIl} \left(\frac{l^2}{3} - a'l + \frac{2}{3}a'^2\right)$  in (208), we have

$$y = \frac{Wa'}{6EIl} [(l^2 - a'^2)(l - x) - (l - x)^3], \quad (212)$$

which is the deflection when x > a'; that is, at any point between the weight W and the right-hand support.

EXAMPLE. — Take a beam of pine weighing 40 pounds per cubic foot, of rectangular cross-section. Depth =  $h = 18\frac{1}{2}$  inches, breadth = b = 15 inches, length =  $12\frac{1}{2}$  feet = 150

inches. Call E = 1,680,000 pounds per square inch,  $I = \frac{15 \times 18.5^3}{12} = 7,914.53125$ ,  $EI = 168 \times 79,145,312.5$ ; beam's own weight per inch of length  $= w = \frac{18.5 \times 15 \times 40}{12^3} = 6.4236\frac{1}{9}$  pounds. Deflection due beam's own weight, lw, at a point 48 inches from one end is, by (205),

$$y = \frac{6.4236\frac{1}{9}}{24 \times 168 \times 79145312.5} (48^4 - 2 \times 150 \times 48^3 + 150^3 \times 48)$$
  
= 0.0027 inch.

Deflection at centre, from beam's weight, by (206), is

$$D = \frac{.5 \times 6.4236\frac{1}{8} \times 150^4}{384 \times 168 \times 79145312.5} = 0.0032 \text{ inch,}$$

which is a maximum.

Deflection at the point of application, due weight W = 17,935 pounds placed a' = 48 inches from end of beam, is, by (210),

$$D = \frac{17935 \times 48^2}{3 \times 168 \times 79145312.5} \times (150 - 48)^2 = 0.07185 \text{ inch.}$$

Deflection at the centre when  $W_1 = 17,935$  pounds is placed 48 inches from one end, is found by equation (212), making  $x = \frac{1}{2}l$ ,

$$y = \frac{17935 \times 48}{12 \times 168 \times 79145312.5} \left( 150^2 - 48^2 - \frac{150^2}{4} \right) = 0.078617 \text{ inch.}$$

And when W is at the centre, the central deflection is, from (211),

$$D = \frac{.17935 \times 150^{3}}{48 \times 168 \times 79145312.5} = 0.0948 \text{ inch.}$$

Add deflection due beam's own weight for total maximum deflection = 0.098 inch.

74. Deflection due a Partial Load, wb, Uniformly Distributed Continuously over the Length, b, beginning at the Horizontal Distance, a, from the Origin,  $O_1$ , or Left End of the Beam, Fig. 9.

To find this deflection, we use, when x < a, equations (53) and (187), giving

$$EI\frac{d^2y}{dx^2} = -w'b'\frac{l - (a + \frac{1}{2}b)}{l}x = -\epsilon x \text{ (say)}.$$

Let  $\alpha$  be the angle of inclination, or slope, of the beam at the distance a from the left end; then integrating, with the condition that  $\frac{dy}{dx} = \tan \alpha$  when x = a,

$$\therefore EI\left(\frac{dy}{dx}-\tan\alpha\right)=-\frac{8}{2}(x^2-a^2).$$

Again, y = 0 when x = 0,

$$\therefore EI(y-x\tan\alpha)=-\frac{\varepsilon}{2}\left(\frac{x^3}{3}-a^2x\right). \tag{213}$$

Let  $y = y_1$  when x = a,

$$\therefore EI(y_1 - a \tan \alpha) = \frac{\epsilon a^3}{3} = \frac{w'a^3b}{3l}(l - a - \frac{1}{2}b). \quad (214)$$

But when x > a and < (a + b), equations (55) and (187) are to be employed, yielding

$$EI\frac{d^2y}{dx^2} = -\epsilon x + \frac{w'}{2}(x^2 - 2ax + a^2)$$
$$= \frac{w'}{2}x^2 - (\epsilon + aw')x + \frac{a^2w'}{2}.$$

And, if  $\beta$  is the angle of inclination at the distance (a + b) from the origin, we may integrate as follows:

$$\frac{dy}{dx} = \tan \alpha \text{ when } x = a,$$

$$\therefore EI\left(\frac{dy}{dx} - \tan\alpha\right)$$

$$= \frac{w'\left(\frac{x^3 - a^3}{3}\right) - \frac{s + aw'}{2}(x^2 - a^2) + \frac{a^2w'}{2}(x - a).$$

 $y = y_1$  when x = a,

$$EI[y-y_1-(x-a)\tan\alpha] = \frac{w'}{6} \left\{ \frac{x^4-a^4}{4} - a^3(x-a) \right\}$$

$$-\frac{\epsilon + aw'}{2} \left\{ \frac{x^3-a^3}{3} - a^2(x-a) \right\} + \frac{a^2w'}{2} \left\{ \frac{x^2-a^2}{2} - a(x-a) \right\}. (215)$$

Let  $y = y_2$  when x = a + b, Fig. 9; so that, after reducing, (215) becomes

$$EI(y_2 - y_1 - b \tan \alpha) = \frac{w'b^3}{l} \left( \frac{a^2}{2} + \frac{5ab}{12} + \frac{b^2}{12} - \frac{al}{2} - \frac{bl}{8} \right). \quad (216)$$

Or, we may integrate in a different manner; first, with the condition  $\frac{dy}{dx} = \tan \beta$  when x = a + b,

$$\therefore EI\left(\frac{dy}{dx} - \tan\beta\right) \\ = \frac{w'}{2} \left\{ \frac{x^3 - (a+b)^3}{3} \right\} - \frac{s + aw'}{2} \left\{ x^2 - (a+b)^2 \right\} + \frac{a^2w'}{2} (x - a - b).$$

Also  $y = y_2$  when x = a + b,

$$\therefore EI[y-y_2-(x-a-b)\tan\beta]$$

$$= \frac{w'}{6} \left\{ \frac{x^4 - (a+b)^4}{4} - (a+b)^3 (x-a-b) \right\}$$

$$- \frac{e+aw'}{2} \left\{ \frac{x^3 - (a+b)^3}{3} - (a+b)^2 (x-a-b) \right\}$$

$$+ \frac{a^2w'}{2} \left\{ \frac{x^2 - (a+b)^2}{2} - (a+b)(x-a-b) \right\}. \tag{217}$$

But in (217)  $y = y_1$  when x = a; therefore, after reducing, we have

$$EI(y_1 - y_2 + b \tan \beta) = \frac{w'b^3}{l} \left( \frac{a^2}{2} + \frac{7ab}{12} + \frac{b^2}{6} - \frac{al}{2} - \frac{5bl}{24} \right). \quad (218)$$

For the remaining part of the beam, Fig. 9, that is, when x > (a + b), equations (57) and (187) give

$$EI\frac{d^2y}{dx^2} = -\epsilon x + w'b(x - a - \frac{1}{2}b) = (w'b - \epsilon)x - w'b(a + \frac{1}{2}b).$$

$$\frac{dy}{dx} = \tan \beta \text{ when } x = a + b,$$

$$\therefore EI\left(\frac{dy}{dx} - \tan\beta\right)$$

$$= \frac{w'b - \varepsilon}{2} \left[x^2 - (a+b)^2\right] - w'b(a + \frac{1}{2}b)(x - a - b).$$

y = 0 when x = l,

$$EI[y - (x - l) \tan \beta]$$

$$= \frac{w'b - \epsilon}{2} \left\{ \frac{x^3 - l^3}{3} - (a + b)^2 (x - l) \right\} - \frac{w'b(a + \frac{1}{2}b)}{2} (x^2 - l^2)$$

$$+ w'b(a + b) (a + \frac{1}{2}b) (x - l), \quad (219)$$

which becomes, if we put  $y_2$  for  $y_2$ , and a + b for  $x_2$ , and reduce,

$$EI[y_2 - (a+b-l)\tan\beta] = \frac{w'b}{3l}(l-a-b)^3(a+\frac{1}{2}b). \quad (220)$$

From equations (214), (216), (218), and (220), we may now determine the four quantities,  $\tan \alpha$ ,  $\tan \beta$ ,  $y_1$ ,  $y_2$ , so that they can be eliminated from (213), (215), (217), and (219).

$$\tan \alpha = \frac{w'b}{Ell} \left( \frac{2a^3}{3} + \frac{a^2b}{2} + \frac{ab^2}{6} + \frac{b^3}{24} - a^2l - \frac{abl}{2} - \frac{b^2l}{6} + \frac{al^2}{3} + \frac{bl^2}{6} \right), \quad (221)$$

$$\tan \beta = \frac{w'b}{EIl} \left( \frac{2a^3}{3} + \frac{3a^2b}{2} + \frac{7ab^2}{6} + \frac{7b^3}{24} - a^2l - \frac{3abl}{2} - \frac{b^2l}{2} + \frac{al^2}{3} + \frac{bl^2}{6} \right), \quad (222)$$

$$y_1 = \frac{w'ab}{EIl} \left( \frac{a^3}{3} + \frac{a^2b}{3} + \frac{ab^2}{6} + \frac{b^3}{24} - \frac{2a^2l}{3} - \frac{abl}{2} - \frac{b^2l}{6} + \frac{al^2}{3} + \frac{bl^2}{6} \right), \quad (223)$$

$$y_{2} = \frac{w'b}{EIl} \left( \frac{a^{4}}{3} + a^{3}b + \frac{7a^{2}b^{2}}{6} + \frac{5ab^{3}}{8} + \frac{b^{4}}{8} - \frac{2a^{3}l}{3} - \frac{3a^{2}bl}{2} - \frac{7ab^{2}l}{6} - \frac{7b^{3}l}{24} + \frac{a^{2}l^{2}}{3} + \frac{abl^{2}}{2} + \frac{b^{2}l^{2}}{6} \right). \quad (224)$$

We have, then, from (213), where x is not greater than a,

$$y = \frac{w'b}{6EIl}(l - a - \frac{1}{2}b)(3a^2x - x^3) + x\tan\alpha, \quad (225)$$

which is the deflection due w/b at any point between the origin and the beginning of the partial continuous uniform load w/b, Fig. 9.

For the uniformly loaded part, b, of the beam, we find, from (215),

$$y = \frac{w'}{2EI} \left\{ \frac{x^4 - a^4}{12} - \frac{a^3}{3}(x - a) - \left[ \frac{b(l - a - \frac{1}{2}b)}{l} + a \right] \left[ \frac{x^3 - a^3}{3} - a^2(x - a) \right] + a^2 \left[ \frac{x^2 - a^2}{2} - a(x - a) \right] \right\} + y_1 + (x - a) \tan \alpha, \quad (226)$$

which is the deflection due w'b at any point of the loaded portion b, since x is here not less than a, nor greater than a + b, Fig. 9.

Equation (219) gives the deflection for the remaining part of the beam, that is, where x is not less than a + b; and we find

$$y = \frac{w/b(a + \frac{1}{2}b)}{2EI} \left\{ \frac{x^3 - l^3}{3l} - \frac{(a+b)^2(x-l)}{l} - (x^2 - l^2) + 2(a+b)(x-l) \right\} + (x-l)\tan\beta, \quad (227)$$

which is the deflection due wb at any point between the right-hand end of the beam and the load wb, Fig. 9.

If  $x = a = b = \frac{1}{4}l$  in (225) or (226), we have the central deflection when one-half of the beam is uniformly loaded con-

. by (206) for

or (227) by e-half load is effection due centre of the cal line; that - 2a. Then, deflection a

ection of the load has any 25), (226), or allr ender y a pint so found, ection, may be made m continuous by assigning he deflection load wb.

),

 ${}^{2}l^{2})x$ , (228)

which is the deflection at any point of the loaded part of the beam, where x is not greater than b.

Also, if a = 0,

$$\tan \beta = \frac{w'b^2}{2AEI}(7b^2 - 12bl + 4l^2),$$

and (227) becomes

$$y = \frac{w'b^2}{24EIl} \Big\{ 2(x^3 - l^3) - 6(x^2 - l^2)l + (b^2 + 4l^2)(x - l) \Big\}, (229)$$

which is the deflection at any point of the unloaded part of the beam, where x is not less than b.

Examples. — Partial uniform continuous load, wb. Wroughtiron 15-inch I-beam. Length 30 feet = 360 inches = l. Moment of inertia  $I = \frac{1}{12}(b_2h_2^3 - b_1h_1^3)$ , by Table III. 5.

Let  $h_2 = 15$  inches,  $b_2 = 5\frac{3}{8}$  inches,  $h_1 = 12\frac{3}{4}$  inches,  $b_1 = 4\frac{3}{4}$  inches, Fig. 91; putting  $h_2$  for h, and  $h_2$  for h, to avoid confusion here. Beam supported at ends. Load w' = 75 pounds per inch of the length h,

$$\therefore I = \frac{1}{12}(5.375 \times 15^3 - 4.75 \times 12.75^3) = 691.$$

Take E = 24,000,000, Table II.,

$$EI = 16584000000$$

all dimensions to be in inches. Let the load cover the first 10 feet of the beam.

1st, What is the deflection at the end of the load?

We have  $x = b = \frac{1}{8}l = 120$  inches; and (228) applies, giving .

$$y = \frac{75 \times 360^4}{24 \times 16584000000} \left( \frac{1}{81} - 4 \times \frac{1}{3} \times \frac{5}{6} \times \frac{1}{27} + \frac{1}{243} - 4 \times \frac{1}{81} + 4 \times \frac{1}{27} \right)$$
  
= 0.23444 inch.

2d, What is the deflection at the centre of the beam?

We have from (229), if  $x = \frac{1}{2}l$ , and  $b = \frac{1}{2}l$ ,

$$y = \frac{75 \times \frac{1}{9} \times 360^{4}}{24 \times 16584000000} \left\{ -2 \times \frac{7}{8} + 6 \times \frac{3}{4} - \frac{1}{2} \left( \frac{1}{9} + 4 \right) \right\}$$
  
= 0.24421 inch.

3d, What is the deflection 10 feet from the unloaded end of the beam?

Here we use (229) also, putting  $x = \frac{2}{3}l$ , and  $b = \frac{1}{3}l$ ,

$$y = \frac{75 \times \frac{1}{9} \times 360^4}{24 \times 16584000000} \left\{ -2 \times \frac{19}{27} + 6 \times \frac{5}{9} - \frac{1}{3} \left( \frac{1}{9} + 4 \right) \right\}$$
  
= 0.19176 inch.

4th, Suppose it is now required to find the point of greatest deflection due this same load of 75 pounds to the inch on 10 feet of one end of the beam.

Differentiating (229), we find, since  $b = \frac{1}{8}l$ ,

$$\frac{dy}{dx} = 6x^3 - 12lx + \frac{37}{9}l^3,$$

omitting constants. Putting this value of  $\frac{dy}{dx}$  equal to zero, we

have x = 0.43892l, which is the point of greatest on; and, by placing this value of x in (229), there y = 0.24847 inch, which is the greatest deflection of m due this load along one end.

But if this same load be moved along to the centre, so have  $a = \frac{1}{8}l = b$ , we find the greatest deflection the

load can produce, by putting  $x = \frac{1}{2}l$  in equation (226), where  $y_1$  becomes  $= \frac{11w'l^4}{1944EI}$ , and  $\tan \alpha = \frac{7w'l^3}{648EI}$ , from (223) and (221). Thus (226) becomes

$$y = \frac{75 \times 360^4}{2 \times 16584000000} \left\{ \frac{I}{I2} \left( \frac{I}{I6} - \frac{I}{8I} \right) - \frac{I}{8I} \left( \frac{I}{2} - \frac{I}{3} \right) - \left[ \frac{I}{3} \left( I - \frac{I}{3} - \frac{I}{6} \right) + \frac{I}{3} \right] \left[ \frac{I}{3} \left( \frac{I}{8} - \frac{I}{27} \right) - \frac{I}{9} \left( \frac{I}{2} - \frac{I}{3} \right) \right] + \frac{I}{9} \left[ \frac{I}{2} \left( \frac{I}{4} - \frac{I}{9} \right) - \frac{I}{3} \left( \frac{I}{2} - \frac{I}{3} \right) \right] \right\} + y_I + \left( \frac{I}{2} - \frac{I}{3} \right) \tan \alpha.$$

Deflection at centre = y = 0.50063 inch,  $\frac{1}{8}w'l$  central; deflection at centre = y = 0.24421 inch,  $\frac{1}{8}w'l$  at either end;

$$\therefore$$
  $y = 0.50063 + 2 \times 0.24421 = 0.98905 inch,$ 

equals deflection at centre when the given 15-inch I-beam of 30 feet between supports is loaded with 13.5 tons, uniformly distributed continuously. And this result accords exactly with that given by (206); thus,

$$D = \frac{5 \times 75 \times 360^4}{384 \times 16584000000} = 0.98905 \text{ inch;}$$

where, as in other values of the deflection, we have retained several unnecessary decimals, in order to test the accuracy of the solutions.

6th, If this beam is half loaded with 75 pounds to the inch, we have in (228), for the deflection at the centre,  $x = b = \frac{1}{2}l$  = 180 inches; and

$$y = \frac{75 \times 360^4}{24 \times 16584000000} \left\{ \frac{1}{16} - 4 \times \frac{1}{2} \times \frac{3}{4} \times \frac{1}{8} + \left( \frac{1}{16} - 4 \times \frac{1}{8} + 4 \times \frac{1}{4} \right) \right\}$$
  
= 0.494525 inch,

which is half the deflection due the fully loaded beam, as just found.

7th, The maximum deflect of the beam is found, both in prentiating (228), putting  $\frac{dy}{dx} = c$  equation, putting  $b = \frac{1}{2}l$ , l = factors,

$$\frac{dy}{dx} = 4x^3 - \frac{9}{2} \times 364$$

$$\therefore x^3 - 405x^2$$

Solving this equation by Havalues,

$$x = 16$$

$$x = 35$$

$$x = -11$$

But, since x must be positive the value here sought is

$$x = 10$$

retaining decimals. Hence the within the loaded part, and is inches from the centre of the l

Putting this value of x is deflection y = 0.49855 inch.

8th, The beam's own wei wrought-iron five-eighteenths parea of cross-section =  $\frac{5}{18}(b_2)$  4.75 × 12.75) = 5.573 pound (206), gives the deflection at weight = 0.07349 inch; so that the fully loaded beam is 0.9890

76. To find the Deflection at any Point, x, due any Number,  $(r_1 - r_2)$ , Equal Weights, W, placed at Equal Intervals, c, along the Beam, the First Weight being Distant by One or More Entire Intervals, c, from the Origin or Left End of the Beam. — For all the weights,  $(r - r_2)$  in number, between the left end and the point x, we use equation (212), which reduces to

$$y = \frac{W}{6EIl}[(l^2a' - a'^3)(l - x) - a'(l - x)^3].$$

Now let a' take the successive values  $c(r_2 + 1)$ ,  $c(r_2 + 2)$ ,  $c(r_2 + 3)$ , ...  $c(r_2 + r - r_2)$ , and we have, by summing,

$$\Sigma a' = c(\overline{r_2 + 1} + \overline{r_2 + 2} + \overline{r_2 + 3} + \dots + r)$$

$$= \frac{1}{2}c(r - r_2)(r + r_2 + 1),$$

$$\Sigma(a'^3) = c^3(\overline{r_2 + 1}^3 + \overline{r_2 + 2}^3 + \overline{r_2 + 3}^3 + \dots + r^3)$$

$$= \frac{1}{4}c^3[r^2(r + 1)^2 - r_2^2(r_2 + 1)^2].$$

Hence (212) becomes

$$y = \frac{W}{24EIl} \Big\{ \{ 2c(r - r_2)(r + r_2 + 1)l^2 - c^3[r^2(r + 1)^2 - r_2^2(r_2 + 1)^2] \} (l - x) - 2c(r - r_2)(r + r_2 + 1)(l - x)^3 \Big\}, \quad (230)$$

which is the deflection due  $r - r_2$  equal weights, W, at any point, x, between the  $r^{th}$  weight and the right end of the beam;  $r_2$  being the number of full intervals vacant at the left end, and x being not less than cr.

For the  $r_1 - r$  equal weights between the point x and the right end of the beam, we employ (209), which reduces to

$$y = \frac{W}{6EIl}[(2l^2a' - 3la'^2 + a'^3)x + (a' - l)x^3].$$

If in this equation a' takes the successive values c(r + 1), c(r + 2), c(r + 3), c(r + 4), . . .  $c(r + r_1 - r)$ , then, by summing, we find

$$\Sigma a' = c[(r+1) + (r+2) + (r+3) + \dots + r_1]$$

$$= \frac{1}{3}c(r_1 - r)(r_1 + r + 1),$$

$$\Sigma a'^2 = c^2[(r+1)^2 + (r+2)^2 + (r+3)^2 + \dots + r_1^2]$$

$$= \frac{1}{6}c^2[r_1(r_1 + 1)(2r_1 + 1) - r(r+1)(2r+1)],$$

$$\Sigma a'^3 = c^3[(r+1)^3 + (r+2)^3 + (r+3)^3 + \dots + r_1^3]$$

$$= \frac{1}{6}c^3[r_1^2(r_1 + 1)^2 - r^2(r+1)^2],$$

$$\Sigma a'^6 = r_1 - r.$$

Hence, for this case, (209) becomes

$$y = \frac{W}{24EII} \Big\{ \{4I^{2}c(r_{t} - r)(r_{t} + r + 1) - 2Ic^{2}[r_{1}(r_{1} + 1)(2r_{1} + 1) - r(r + 1)(2r + 1)] + c^{3}[r_{t}^{2}(r_{t} + 1)^{2} - r^{2}(r + 1)^{2}] \}x + [2c(r_{t} - r)(r_{t} + r + 1) - 4(r_{t} - r)I]x^{3} \Big\}, \quad (231)$$

which is the deflection due the  $r_1 - r$  equal weights on the beam at any point, x, between the left end and the (r + 1)<sup>th</sup> weight; x not being greater than c(r + 1).

Adding the deflections given by (230) and (231), and calling their sum also y, we have

$$y = \frac{W}{24Ell} \Big\{ \Big[ 2c(r_1 - r_2)(r_1 + r_2 + 1) - 4(r_1 - r)l \Big] x^5 \\ -6c(r - r_2)(r + r_2 + 1)lx^2 \Big\} + \Big\{ 4l^2c(r_1 - r_2)(r_1 + r_2 + 1) + c^3 \Big[ r_1^2(r_1 + 1)^2 - r_2^2(r_2 + 1)^2 \Big] \\ - 2lc^2 \Big[ r_1(r_1 + 1)(2r_1 + 1) - r(r + 1)(2r + 1) \Big] \Big\} x \\ - c^3 l \Big[ r^2(r + 1)^2 - r_2^2(r_2 + 1)^2 \Big] \Big\}, \quad (232)$$

which is the deflection at any point, x, due the  $r_1 - r_2$  equal weights, W; where  $r_1$  denotes the number of intervals between the last weight and the left end of the beam, r a number of full intervals not less than  $r_2$ , the number of unloaded intervals at the left end of the beam, nor greater than  $r_1$ .

If in (232) we put  $c = l \div n$ ,  $x = \frac{1}{2}l$ ,  $r_2 = 0$ ,  $r_1 = n - 1$ , and  $r = \frac{1}{2}n$  when n is even, but  $r = \frac{1}{2}(n - 1)$  when n is odd, we shall find

$$D = \frac{Wl^{3}}{384EIn} (5n^{2} - 4), n \text{ even,}$$

$$= \frac{Wl^{3}}{384EIn^{3}} (5n^{4} - 4n^{2} - 1), n \text{ odd,}$$
(233)

which is the deflection at the centre due the  $r_1 = n - 1$  equal weights, W, covering the beam of n equal intervals  $(l \div n)$ .

EXAMPLES. — Let us take the same 15-inch I-beam we employed in the examples of article 75, for which I=691, E=24,000,000, l=360 inches. Take 3 weights of 4,500 pounds each, placed at intervals of 60 inches, beginning at one end of the beam; then the deflection at the centre is given by (230) if we put W=4,500, l=360,  $c=\frac{1}{6}l=60$ ,  $r_2=0$ , r=3,  $x=\frac{1}{2}l$ , EI=16,584,000,000. Thus,

$$y = \frac{4500 \times 360^{3}}{24 \times 16584000000} \left( 2 \times \frac{1}{6} \times 3 \times 4 \times \frac{1}{2} - \frac{1}{6^{3}} \times 9 \right)$$

$$\times 16 \times \frac{1}{2} - 2 \times \frac{1}{6} \times 3 \times 4 \times \frac{1}{8} \right) = 0.6154 \text{ inch.}$$

If 2 more equal weights are added at the same interval, so as to cover the beam, the central deflection due these last 2 is, by (231), where  $r_1 = 5$ , r = 3,

$$y = \frac{4500 \times 360^{3}}{24 \times 16584000000} \left\{ 4 \times \frac{1}{6} \left( 25 - 9 + 5 - 3 \right) \frac{1}{2} \right.$$

$$- 2 \times \frac{1}{36} \left( 5 \times 6 \times 11 - 3 \times 4 \times 7 \right) \frac{1}{2} + \frac{1}{6^{3}} \left( 25 \times 36 - 9 \times 16 \right)$$

$$+ 2 \times \frac{1}{6} \left( 25 - 9 + 5 - 3 \right) \frac{1}{8} - 4 \times 2 \times \frac{1}{8} \right\} = 0.3517 \text{ inch.}$$

If we compute the central deflection due these 5 equal weights by (233), we have n = 6, and

$$D = \frac{4500 \times 360^3}{384 \times 16584000000} \left( \frac{5 \times 36 - 4}{6} \right) = 0.9671 \text{ inch,}$$

which is the sum of the deflections found by (230) and (231).

Again, if there are 8 weights upon the beam, each equal to W = 3,000 pounds, at intervals of 40 inches, we have n = 9, l = 360 inches; and (233) gives the central deflection,

$$D = \frac{3000 \times 360^3}{384 \times 16584000000} \left( \frac{5 \times 9^4 - 4 \times 9^2 - 1}{9^3} \right) = 0.97926 \text{ inch.}$$

In these examples the weight has been purposely chosen equal to 75 pounds to the inch for the entire length of the beam, except a half-interval,  $(/ \div 2n)$ , at each end; so that we may compare the results with the central deflection of the same beam, computed by (206) for the continuous uniform load of 75 pounds to the inch, which deflection we have found to be 0.98905 inch.

Now it will be found that the central deflection due the discontinuous load, (n-1)W, at equal intervals,  $(l \div n)$ , will be less than that due the continuous uniform load, lw, until n becomes infinite, and  $W = \frac{lw}{n}$  infinitesimal, when (233) becomes

The greatest difference between the central deflections of these two loads, (n-1)W and lw, manifestly occurs when n=2; that is, when there is but one weight, and that at the centre, and equal to  $W=\frac{lw}{2}$ . Equation (233) then becomes

 $D = \frac{4\pi v l^4}{384EI}$ , which is four-fifths of the deflection due lw continuously distributed uniformly, as shown by (206).

From these considerations it appears, that, in practice, the formulæ found in article 75, for a uniform continuous load, are applicable to a uniform load distributed, as above, discontinuously, or by panel weights, each equal to  $(lw \div n)$ , provided n is large.

But in any case, whether there be many or few intervals, we may find, by means of equation (232), the greatest deflection due any partial or complete load of equal panel weights, W, and the point where it occurs.

For this purpose, differentiate (232) with respect to x, and put  $\frac{dy}{dx} = 0$ . This gives

$$[6c(r_1-r_2)(r_1+r_2+1)-12(r_1-r)l]x^2$$

$$-12c(r-r_2)(r+r_2+1)lx$$

$$+4l^2c(r_1-r_2)(r_1+r_2+1)+c^3[r_1^2(r_1+1)^2-r_2^2(r_2+1)^2]$$

$$-2lc^2[r_1(r_1+1)(2r_1+1)-r(r+1)(2r+1)]=0, (234)$$

from which we find

$$x = A \pm \sqrt{A^2 + B}, \qquad (235)$$

where

$$A = \frac{c(r-r_2)(r+r_2+1)l}{c(r_1-r_2)(r_1+r_2+1)-2(r_1-r)l},$$

and

$$B = \frac{2lc^2[r_1(r_{1}+1)(2r_{1}+1)-r(r+1)(2r+1)]-4l^2c(r_{1}-r_{2})(r_{1}+r_{2}+1)-c^3[r_{1}^2(r_{1}+1)^2-r_{2}^2(r_{2}+1)^2]}{6c(r_{1}-r_{2})(r_{1}+r_{2}+1)-12(r_{1}-r)l}.$$

Now put cr for x in (234); and find r by trial, easily, since it is an integer, and the point of greatest deflection is approximately known, by inspection, for any given load. Then, having r, compute x in (235), after which the greatest deflection, y, may be found by equation (232).

EXAMPLE 1. — Given the wrought-iron I-beam of article 75, where I = 691, E = 24,000,000, l = 360 inches; and let there be upon it 5 weights of 4,500 pounds each, at equal intervals of  $c = \frac{1}{6}l = 60$  inches. We then have W = 4,500,  $r_1 = 5$ ,  $r_2 = 0$ ; and, by putting cr for x in equation (234), we find

$$2r^3 - 18r^2 + r + 105 = 0.$$

By trial, we see that r=3, as we should also infer from the symmetrical load. Making r=3 in (235), there results  $x=\frac{1}{2}l$ . This value of x placed in (232) gives the deflection y=0.9671 inch, as by (233).

EXAMPLE 2. — If on this same beam we have 4,500 pounds at the end of the second and third intervals, we have W = 4,500,  $r_1 = 3$ ,  $r_2 = 1$ ,  $c = \frac{1}{6}l = 60$  inches; and, by putting cr for x in equation (234), we find

$$6r^3 - 48r^2 + 39r + 143 = 0$$

where, by trial, we see that r lies between 2 and 3. Making r = 2 in (235), we find x = 0.48141l. With this value of x, (232) gives the maximum deflection due the 2 given weights, y = 0.48934 inch.

Example 3. — If these 2 equal weights are at the end of the third and fourth intervals, then  $r_1 = 4$ ,  $r_2 = 2$ , c = 60; and we shall find r = 3, x = 0.51859l, and the maximum deflection, as before, y = 0.48934 inch.

## SECTION 3.

The Influence of Fixed Ends upon the Deflection of a Beam of Uniform Cross-Section, Supported at its Two Extremities, which are Assumed to be Level, and One or Both of Them Fixed Horizontally or Otherwise. Determination of the End Moments and Points of Contrary Flexure.

77. The influence of the end couples upon the moments due the other forces has already been found, by equation (93), to be

$$M_x = \frac{M_2 - M_1}{l}x + M_1,$$

where  $M_1$  is the left end moment, and  $M_2$  the right end moment, of the fixed beam, Fig. 12.

Wherefore, to find the deflection due these end couples, (187) becomes

$$EI\frac{d^2y}{dx^2} = \frac{M_1 - M_2}{l}x - M_1;$$

giving the first member the positive sign, since  $M_1$  and  $M_2$  are here assumed to have a tendency to deflect the beam upward, and are negative relatively to the moments tending to deflect it downwards.

If  $\alpha$  = the slope of the beam at the centre, then  $\frac{dy}{dx} = \tan \alpha$  when  $x = \frac{1}{2}l$ , and the first integration yields

$$EI\left(\frac{dy}{dx}-\tan\alpha\right)=\frac{M_1-M_2}{2l}\left(x^2-\frac{l^2}{4}\right)-M_1(x-\frac{1}{2}l).$$

Again, since y = 0 when x = 0,

$$\therefore EI(y-x\tan\alpha)=\frac{M_1-M_2}{2l}\left(\frac{x^3}{3}-\frac{l^2}{4}x\right)-M_1\left(\frac{x^2}{2}-\frac{l}{2}x\right).$$

$$\tan\alpha = -\frac{(M_1 - M_2)I}{24EI}.$$

$$\frac{M_2}{M_2}(x^3-l^2x)-3M_1(x^2-lx)\bigg\},\quad (236)$$

on due the end moments in terms of these ents. Now, since (236) has been found e ends of the beam tangent to the line vo points of support, we may suppose  $M_{\bullet}$  be equal to each other.

$$y = \frac{M_3}{6El} \left( lx - \frac{x^3}{l} \right). \tag{237}$$

$$y = \frac{M_1}{6EI} \left( \frac{x^3}{I} - 3x^2 + 2Ix \right). \tag{238}$$

$$y = \frac{M}{2EI}(lx - x^2).$$
 (239)

nine the end moments in particular cases, particular mode of loading.

Lous and Uniform throughout, = w per - If we add equations (236) and (205), callnave

$$t^3 + l^3x$$

$$\frac{f_2}{2}(4x^3-4l^3x)-12M_1(x^2-lx)\bigg\},\qquad (240)$$

on at any point of the uniformly loaded

If the ends are both fixed horizontally (that is, if the tangent to the curve is horizontal at each end of the level beam), we must have  $M_1 = M_2$ , since the load is uniform. And, differentiating (240),

$$\frac{dy}{dx} = \frac{1}{24EI} \left\{ w(4x^3 - 6lx^2 + l^3) + \frac{M_1 - M_2}{l} (12x^2 - 4l^2) - 12M_1(2x - l) \right\}.$$

But, now,  $\frac{dy}{dx} = 0$  when x = 0,

$$\therefore wl^3 + 12M_1l = 0,$$

$$M_1 = M_2 = -\frac{1}{12}wl^2. \tag{241}$$

Putting this value of the end moments into (240), there results

$$y = \frac{w}{24EI}(I-x)^2x^2, \qquad (242)$$

which is the deflection at any point, x, of a beam with ends fixed horizontally, under a confinuous uniform load, w, per unit of length, l.

If  $x = \frac{1}{2}l$ , we have the central deflection

$$D=\frac{wl^4}{384EI},\tag{243}$$

which is one-fifth that due the same load on the same beam with its ends not fixed, as given by (206).

Since  $M_1 = M_2$ , the total moment due lw at any point, is, by (49) and (93),

$$M_x = \frac{1}{2}w(l-x)x + M_1 = w\left(\frac{lx-x^2}{2} - \frac{1}{12}l^2\right).$$

And, if we put this moment  $M_x = 0$ , we shall have x representing the distance from the left end of the beam to the *points of* contrary flexure, as those points are called where the curvature changes from convex to concave upward.

Therefore

$$x^{2} - lx + \frac{1}{8}l^{2} = 0,$$
  
 $x = 0.21133l$  or  $0.78867l$ . (244)

79. But if the right end of the beam is fixed horizontally, while the left end is not fixed at all, we have  $M_1 = 0$ , and (240) becomes

$$y = \frac{1}{24EI} \left\{ w(x^4 - 2lx^3 + l^3x) - 4M_2 \frac{x^3 - l^2x}{l} \right\}; (245)$$

and

$$\frac{dy}{dx} = \frac{1}{24EI} \left\{ w(4x^3 - 6lx^2 + l^3) - \frac{M_2}{l} (12x^4 - 4l^2) \right\}$$

equal to 0 when  $x = \lambda$ 

$$\therefore M_2 = -\frac{1}{8} \psi l^2. \tag{246}$$

Hence, from (245),

$$y = \frac{w}{48EI}(2x^4 - 3lx^3 + l^3x), \qquad (247)$$

which is the deflection at any point of a beam horizontally fixed at one end, and simply supported at the other, under a uniform load w per unit of length, l; x to be measured from the unfixed end.

Since, now,  $M_1 = 0$ , the total moment due lw at any point is, from (49) and (93),

$$M_x = \frac{1}{2}w(l-x)x + \frac{M_xx}{l} = w(\frac{1}{2}lx - \frac{1}{2}x^2 - \frac{1}{8}lx).$$

If 
$$M_x = 0$$
,  $x = \frac{3}{4}l$ , (248)

which is the distance of the point of contrary flexure from the free end of the beam, under the load *lw* uniformly distributed continuously.

Examples. — Suppose the wrought-iron 15-inch I-beam 30 feet in length, of the examples in article 75, to be fixed horizontally at both ends, and loaded uniformly with 75 pounds to each inch of its length; what is the deflection 10 feet from either end? We now have I = 691, E = 24,000,000, l = 360,  $x = \frac{1}{8}l$  or  $\frac{2}{8}l$ . Hence (242) gives the deflection

$$y = \frac{75 \times 360^4 \times 4}{24 \times 16584000000 \times 81} = 0.1563$$
 inch.

At the centre, where  $x = \frac{1}{2}l$ , (242) gives

$$y = \frac{75 \times 360^4 \times 1}{24 \times 165840000000 \times 16} = 0.19781$$
 inch,

which is one-fifth of that given by (206) for beam with free ends.

If only one end of the beam is fixed, (247) gives,

When 
$$x = \frac{1}{8}$$
,  $y = \frac{75 \times 360^4 \times 10}{24 \times 16584000000 \times 81} = 0.39074$  inch.  
 $x = \frac{1}{2}$ ,  $y = \frac{75 \times 360^4 \times 1}{24 \times 16584000000 \times 8} = 0.39563$  inch.  
 $x = \frac{2}{8}$ ,  $y = \frac{75 \times 360^4 \times 7}{24 \times 16584000000 \times 81} = 0.27352$  inch.

x = 151.7846 inches, y = 0.41141 inch, a maximum.

80. Deflection of a Beam fixed Horizontally at Both Ends, due to a Concentrated Load, W, placed at the Horizontal Distance a' from the Left End of the Beam.—From equations (40), (187), and (93), we have the total moment due W when x is not greater than a',

$$M_{x} = -EI\frac{d^{2}y}{dx^{2}} = W\frac{l-a'}{l}x - \frac{M_{1}-M_{2}}{l}x + M_{1}, \quad (249)$$

$$EI\frac{d^{2}y}{dx^{2}} = \frac{1}{l}[W(a'-l) + M_{1} - M_{2}]x - M_{1}.$$

Integrating, as in article 73,  $\frac{dy}{dx} = \tan \alpha$  when x = a',

$$EI\left(\frac{dy}{dx} - \tan \alpha\right) = \frac{W(a'-l) + M_1 - M_2}{2l}(x^2 - a'^2) - M_1(x - a'). (250)$$

Again, y = 0 when x = 0,

$$EI(y - x \tan \alpha) = \frac{W(a' - l) + M_1 - M_2(x^3 - a'^2x) - M_1(x^2 - a'x)}{2l}. \quad (251)$$

But when x is not less than a', use (43) with (93) and (187), giving

$$M_x = -EI\frac{d^2y}{dx^2} = \frac{Wa'}{l}(l-x) - \frac{M_1 - M_2}{l}x + M_1, \quad (252)$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{l}(Wa' + M_1 - M_2)x - (Wa' + M_1).$$

$$\frac{dy}{dx} = \tan \alpha \text{ when } x = a',$$

$$EI\left(\frac{dy}{dx} - \tan \alpha\right) = \frac{Wa' + M_1 - M_2}{2l}(x^2 - a'^2) - (Wa' + M_1)(x - a'). (253)$$

y = 0 when x = l,

$$EI[y - (x - l) \tan \alpha]$$

$$= \frac{Wa' + M_1 - M_2}{2l} \left\{ \frac{x^3 - l^3}{3} - a'^2(x - l) \right\}$$

$$- (Wa' + M_1) \left\{ \frac{x^2 - l^2}{2} - a'(x - l) \right\}. \quad (254)$$

Now y in (251) is equal to y in (254) when x = a'; therefore, from (251) and (254), we find

$$\tan \alpha = \frac{1}{Ell} \left[ W \alpha' \left( \frac{2}{3} \alpha'^2 + \frac{1}{8} l^2 - \alpha' l \right) + M_1 \left( \frac{1}{3} l^2 + \frac{1}{2} \alpha'^2 - \alpha' l \right) - M_2 \left( \frac{1}{2} \alpha'^2 - \frac{1}{8} l^2 \right) \right]. \quad (255)$$

But in (250) we now have  $\frac{dy}{dx} = 0$  when x = 0,

$$\therefore \tan \alpha = \frac{1}{E l'} \left\{ W a'^2 \frac{a' - l}{2} + M_1 \left( \frac{a'^2}{2} - a' l \right) - M_2 \frac{a'^2}{2} \right\}. \quad (256)$$

Also, in (253)  $\frac{dy}{dx} = 0$  when x = l,

$$\therefore \tan \alpha = \frac{1}{E l l} \left[ W(\frac{1}{2}a'^3 + \frac{1}{2}a'l^2 - a'^2l) + \frac{1}{2}(M_1 - M_2)(a'^2 - l^2) + M_1 l (l - a') \right]. \quad (257)$$

From (255), (256), and (257), we find

$$M_1 = -\frac{W}{l^2}(l-a')^2a',$$
 (258)

$$M_2 = -\frac{W}{l^2}(l-a')a'^2,$$
 (259)

which are the end moments developed by the weight W in any position,  $\alpha'$ .

If the weight W is at the centre,  $a' = \frac{1}{2}l$ , and

$$M_1 = M_2 = -\frac{1}{8}W7. (260)$$

Eliminating  $M_1$ ,  $M_2$ , and  $\tan \alpha$  from equation (251), we find, x not being greater than a',

$$y = \frac{W}{6Ell^3} [(3a'^2l - l^3 - 2a'^3)x^3 + (3a'^3l - 6a'^2l^2 + 3a'l^3)x^2], \quad (261)$$

which is the deflection at any point between the weight W and the left end of the fixed beam with ends horizontal.

Again, eliminating  $M_1$ ,  $M_2$ , and  $\tan \alpha$  from (254), we find, x not being less than  $\alpha'$ ,

$$y = \frac{W}{6Ell^3} [(3a'^2l - 2a'^3)x^3 + (3a'^3l - 6a'^2l^2)x^2 + 3a'^2l^3x - a'^3l^3], \quad (262)$$

which is the deflection due W at any point between W and the right-hand end of the fixed beam.

If  $x = a' = \frac{1}{2}l$ , both (261) and (262) reduce to

$$D=\frac{W7^3}{192EI},\qquad (263)$$

which is the central deflection when the weight W is at the centre of the fixed beam, and is one-fourth of that due the same load on the same beam with its ends not fixed, as seen by equation (211).

To find one point of contrary flexure, we put  $M_x = 0$  in equation (249), and, after eliminating  $M_1$  and  $M_2$ , have

$$x = \frac{a'l(l-a')^2}{2a'^3 - 3a'^2l + l^3}.$$
 (264)

If 
$$a' = \frac{1}{2}l$$
,  $x = \frac{1}{2}l$ . (265)

For the other point of contrary flexure, put  $M_x = 0$  in (252), and the result is

$$x = \frac{l(2l - a')}{3l - 2a'}.$$
 (266)

If 
$$a' = \frac{1}{2}l$$
,  $x = \frac{3}{4}l$ . (267)

From (265) and (267) it appears that when the concentrated load, W, is at the centre of the fixed beam, the points of contrary flexure are each midway between the centre and end of the beam.

81. If the beam is fixed horizontally at the right-hand end, but only supported at the left end, we have  $M_1 = 0$ ; while  $M_2$  may be found from (255) and (257), since the condition that  $\frac{dy}{dx} = 0$  when x = 0, on which (256) depends, does not now exist.

$$M_2 = \frac{W}{2l^2}(a'^2 - l^2)a'. \qquad (268)$$

This value of  $M_2$  placed in either (255) or (257), while  $M_1 = 0$ , gives

$$\tan \alpha = \frac{W}{E l l^3} (a'^3 l^2 - a'^2 l^3 + \frac{1}{4} a' l^4 - \frac{1}{4} a'^5), \quad (269)$$

which is the tangent of the angle of inclination of the beam at any point where the load W may be, while only the right end is fixed; a' to be measured from the free end. With these values of  $M_1$ ,  $M_2$ , and  $\tan \alpha$  substituted in equation (251), we find

$$y = \frac{W}{12E/l^3} [(3a'l^2 - a'^3 - 2l^3)x^3 + (3a'^3l^2 - 6a'^2l^3 + 3a'l^3)x], \quad (270)$$

which is the deflection at any point between the weight W and the unfixed end of the beam, from which end a' and x are to be measured.

In the same manner, from equation (254) we find, x being not less than a',

$$y = \frac{W}{12EIl^3} [(3a'l^2 - a'^3)x^3 - 6a'l^3x^2 + (3a'^3l^2 + 3a'l^4)x - 2a'^3l^3], \quad (271)$$

which is the deflection at any point between the weight W and the horizontally fixed end of the beam;  $\alpha'$  and x being measured from the free end.

If in either (270) or (271) we put  $x = a' = \frac{1}{2}l$ , we have the central deflection

$$D = \frac{7W7^3}{768EI} \tag{272}$$

due the concentrated load W applied at the centre of the beam horizontally fixed at one end.

If we differentiate (270), and put  $\frac{dy}{dx} = 0$ , we shall find

$$x = \pm l \left( \frac{2a'^2l - a'^3 - a'l^2}{3a'l^2 - a'^3 - 2l^3} \right)^{\frac{1}{2}}, \tag{273}$$

which is the point of maximum deflection between the weight W and the free end.

If the weight is at the centre,  $a' = \frac{1}{2}l$ , and

$$x = \pm l\sqrt{\frac{1}{5}} = 0.44721l. \tag{274}$$

In a similar manner, differentiating equation (271), and putting  $\frac{dy}{dx} = 0$ , we find

$$x = \frac{l^3 + a'^2 l}{3l^2 - a'^2}, \tag{275}$$

the point of maximum deflection between W and fixed end, where x cannot be less than a'; that is, a' in this formula cannot be greater than x.

Putting a' for x in (275), we may find easily, by trial, that a' = 0.414213l is the greatest value a' can have in this case of a maximum value of y between the weight W and the horizontally fixed end of the beam.

The point of contrary flexure may be found from (252) by putting  $M_x = 0$ ,  $M_1 = 0$ , and  $M_2$  as in (268). This substitution gives

$$x = \frac{2l^3}{3l^2 - a'^2}. (276)$$

If the weight W is at the centre,  $a' = \frac{1}{2}l$ , and

$$x=\frac{8}{11}l, \qquad (277)$$

which is the distance of the point of contrary flexure from, the free end of the beam.

If a' = 0,  $x = \frac{2}{8}l$ ; and if a' = l, x = l: which are the limits to the range of the point of contrary flexure, for a concentrated load W, on a beam fixed horizontally at one end, and free at the other; x being measured from the free end.

Examples. — Take the 15-inch I-beam of article 75, and suppose it bears a concentrated load W=27,000 pounds, and that both ends are fixed horizontally. We have, as before, I=691, E=24,000,000, l=360 inches.

When W is at the centre, what is the deflection halfway between the centre and either end of the beam?

Put  $a' = \frac{1}{2}l$  and  $x = \frac{1}{4}l$  in (261), or  $a' = \frac{1}{2}l$  and  $x = \frac{3}{4}l$  in (262), and find

$$y = \frac{27000 \times 360^{3}}{6 \times 16584000000} \left[ \left( \frac{3}{4} - 1 - \frac{2}{8} \right) \frac{1}{64} + \left( \frac{3}{8} - \frac{6}{4} + \frac{3}{2} \right) \frac{1}{16} \right] = 0.19781 \text{ inch.}$$

At the centre the deflection is, by (263),

$$D = \frac{27000 \times 360^3}{192 \times 16584000000} = 0.39562 \text{ inch,}$$

which is one-fourth of 1.58248 = the deflection due the same load on the same beam with free ends. And this 1.58248 is, again, eight-fifths of 0.98905, the deflection found by (206) for the same load continuously distributed uniformly over the same beam with free ends.

The points of contrary flexure are given, by (265) and (267), at 90 inches and 270 inches from either end. Now, since the deflection at the quarter-points is just one-half that at the centre, it follows that, in this case, the end of the neutral line, the point of contrary flexure, and the centre are in the same straight line.

When W is at the distance  $a' = \frac{1}{4}l$  from the left end of the beam, what is the maximum deflection?

Differentiating (262), and putting  $\frac{dy}{dx}$  = 0, we find

$$x = 0.4l$$
,  
 $\therefore y = 0.2136 \text{ inch.}$ 

Or, if  $a' = \frac{3}{4}l$ , we find in the same way, from (261),

$$x = 0.6l$$
,  
 $\therefore y = 0.2136 \text{ inch.}$ 

If  $a' = \frac{8}{4}l$ , the points of contrary flexure are, by (264) and (266),  $x = \frac{8}{10}l$ ,  $x = \frac{5}{6}l$ .

But if 
$$a' = \frac{1}{4}l$$
,  $x = \frac{7}{10}l$ .

Let us now suppose that this beam is fixed horizontally at the right-hand end, but is simply supported at the left end. When W = 27,000 pounds is at the centre, what is the deflection at the quarter-points?

Putting  $a' = \frac{1}{2}l$ , and  $x = \frac{1}{2}l$ , we find, from (270),

y = 0.5316 inch.

But if  $a' = \frac{1}{2}l$ , and  $x = \frac{3}{4}l$ , (271) gives

y = 0.3091 inch.

W remaining at the centre, the central deflection is, from (272), D = 0.69234 inch.

Also, from (274) and (270), the maximum deflection due W at the centre is y = 0.70769 inch.

If we place the weight W = 27,000 pounds at the distance a' = 0.414213l from the free end for the maximum value of the deflection y, we shall find, by (275), x = a' = 0.414213l; and from (270) or (271), y = 0.74534 inch, which is the greatest deflection W can produce on this beam, since it is at the point of maximum deflection.

Putting a' = 0.414213l in (276), we find

$$x = 0.707106l$$

the point of contrary flexure when W is at the lowest point of the beam fixed horizontally at one end; x to be measured from the free end.

82. Any Number,  $r_1 - r_2$ , Equal Weights, W, placed at Equal Intervals, c, along the Beam; the First Weight being  $(r_2 + 1)$  Intervals from the Left End, and the Beam being fixed Horizontally at Both Ends.—Let  $r - r_2$  denote the number of equal weights, and r equal the number of full intervals, between the point x and the origin or left end of the beam, Fig. 12; then  $r_1 - r$  = the number of weights between the point x and the right end, if any.

The deflection at the point x due any one of the  $r-r_2$  equal weights, W, is given by equation (262). Let a' in that equation take the successive values  $c(r_2 + 1)$ ,  $c(r_2 + 2)$ ,  $c(r_2 + 3)$ , . . .  $c(r_2 + r - r_2)$ ; then, by summing, we have

$$\Sigma a'^2 = c^2 [(r_2 + 1)^2 + (r_2 + 2)^2 + (r_2 + 3)^2 + \dots + r^2]$$

$$= \frac{c^2}{6} [r(r+1)(2r+1) - r_2(r_2+1)(2r_2+1)],$$

$$\sum a'^3 = c^3 [(r_2 + 1)^3 + (r_2 + 2)^3 + (r_2 + 3)^3 + \dots + r^3]$$

$$= \frac{c^3}{4} [r^2 (r + 1)^2 - r_2^2 (r_2 + 1)^2],$$

which values, put in the place of  $a'^2$  and  $a'^3$  in (262), give

$$y = \frac{W}{6Ell^{3}} \left\{ \left\{ \frac{1}{2}c^{2}[r(r+1)(2r+1)-(r_{2}+1)(2r_{2}+1)r_{2}]l - \frac{1}{2}c^{3}[r^{2}(r+1)^{2}-(r_{2}+1)^{2}r_{2}^{2}] \right\} x^{3} + \left\{ \frac{3}{4}c^{3}[r^{2}(r+1)^{2}-(r_{2}+1)^{2}r_{2}^{2}]l - c^{2}[r(r+1)(2r+1)-(r_{2}+1)(2r_{2}+1)r_{2}]l^{2} \right\} x^{4} + \frac{1}{2}c^{2}[r(r+1)(2r+1)-(r_{2}+1)(2r_{2}+1)r_{2}]l^{3}x - \frac{1}{4}c^{3}[r^{2}(r+1)^{2}-(r_{2}+1)^{2}r_{2}^{2}]l^{3} \right\}, \quad (278)$$

which is the deflection due  $r - r_2$  equal weights at any point, x, between the  $r^{th}$  interval and the right end of the beam having both ends horizontally fixed; x being not less than cr.

If in (278) we make x = cr, and  $r_2 = 0$ , then

$$y = \frac{Wc^3r^2(r+1)}{24Ell^3} [(3r+1)l^3 - 4cr(2r+1)l^2 + c^2r^2(7r+5)l - 2c^3r^3(r+1)], \quad (279)$$

which is the deflection, at the  $r^{th}$  weight, due r equal weights, W, along the left end of the beam at equal intervals, c.

Again, the deflection at the point x due any one of the  $r_1 - r$  equal weights beyond the point x, is given by equation (261).

Let a' in that equation take the successive values c(r + 1), c(r + 2), c(r + 3), . . .  $cr_1$ ; then summing as in article 76, and putting the values of  $\Sigma a'$ ,  $\Sigma a'$ ,  $\Sigma a'^2$ ,  $\Sigma a'^3$ , into equation (261), we find

$$y = \frac{W}{6Ell^{3}} \left\{ \left\{ \frac{1}{2}c^{2} \left[r_{1}(r_{1}+1)(2r_{1}+1)-(r+1)(2r+1)r\right]l - (r_{1}-r)l^{3} - \frac{1}{2}c^{3} \left[r_{1}^{2}(r_{1}+1)^{2}-(r+1)^{2}r^{2}\right] \right\} x^{3} + \left\{ \frac{3}{4}c^{3} \left[r_{1}^{2}(r_{1}+1)^{2}-(r+1)^{2}r^{2}\right]l - c^{2} \left[r_{1}(r_{1}+1)(2r_{1}+1) - (r+1)(2r+1)r\right]l^{2} + \frac{3}{2}c(r_{1}-r)(r_{1}+r+1)l^{3} \right\} x^{2} \right\}, \quad (280)$$

which is the deflection due  $r_1 - r$  equal weights, W, at any point, x, between the (r + 1)<sup>th</sup> interval and the left end of the beam; x being not greater than c(r + 1).

Adding equations (278) and (280), and calling the result y still, we have

$$y = \frac{W}{6Ell^{3}} \left\{ \left\{ \frac{1}{2}c^{2} \left[ r_{1}(r_{1}+1) \left( 2r_{1}+1 \right) - \left( r_{2}+1 \right) \left( 2r_{2}+1 \right) r_{2} \right] \right\} - (r_{1}-r)l^{3} - \frac{1}{2}c^{3} \left[ r_{1}^{2} \left( r_{1}+1 \right) ^{2} - \left( r_{2}+1 \right) ^{2} r_{2}^{2} \right] \right\} x^{3} + \left\{ \frac{3}{4}c^{3} \left[ r_{1}^{2} \left( r_{1}+1 \right) ^{2} - \left( r_{2}+1 \right) ^{2} r_{2}^{2} \right] \right\} - c^{2} \left[ r_{1} \left( r_{1}+1 \right) \left( 2r_{1}+1 \right) - \left( r_{2}+1 \right) \left( 2r_{2}+1 \right) r_{2} \right] l^{2} + \frac{3}{2}c \left( r_{1}-r \right) \left( r_{1}+r+1 \right) l^{3} \right\} x^{2} + \frac{1}{2}c^{2} \left[ r \left( r+1 \right) \left( 2r+1 \right) - \left( r_{2}+1 \right) \left( 2r_{2}+1 \right) r_{2} \right] l^{3} x - \frac{1}{4}c^{3} \left[ r^{2} \left( r+1 \right) ^{2} \right] - \left( r_{2}+1 \right) ^{2} r_{2}^{2} \right) l^{3} \right\}, \quad (281)$$

which is the deflection due all the  $r_1 - r_2$  equal weights at any point, x, between the  $r^{\text{th}}$  and the  $(r + 1)^{\text{th}}$  intervals; x being not less than cr, nor greater than c(r + 1), while r here is not greater than  $r_1$ , nor less than  $r_2$ .

Beam fixed horizontally at both ends. If we now suppose the beam divided into n full intervals, each  $= c = \frac{l}{n}$ , and a

weight, W, at each point of division; and further, if we require the central deflection due such a load, we have  $x = \frac{1}{2}l$ ,  $c = \frac{l}{n}$ ,  $r_1 = n - 1$ ,  $r_2 = 0$ ,  $r = \frac{1}{2}n$  when n is even, but  $r = \frac{1}{2}(n - 1)$  when n is odd.

Placing these values in (281), we obtain

$$D = \frac{W7^{3}n}{384EI'}, \qquad n \text{ even,}$$

$$D = \frac{W7^{3}(n^{4} - 1)}{384EIn^{3}}, n \text{ odd,}$$
(282)

which is the deflection at the centre due the  $r_1 = n - 1$  equal weights, W, covering the beam of n equal intervals,  $\frac{l}{n}$ ; beam fixed horizontally at both ends.

The end moments,  $M_1$ ,  $M_2$ , due a single weight, W, are given by (258) and (259), which reduce to

$$M_1 = -\frac{W}{l^2}(a'l^2 - 2a'^2l + a'^3),$$

$$M_2 = -\frac{W}{l^2}(a'^2l - a'^3).$$

Now let a' take the successive values  $c(r_2 + 1)$ ,  $c(r_2 + 2)$ ,  $c(r_2 + 3)$ , . . .  $c(r_2 + r_1 - r_2)$ , so that we have

$$\Sigma a'^{\circ} = r_1 - r_2,$$

$$\Sigma a' = c(\overline{r_2 + 1} + \overline{r_2 + 2} + \overline{r_2 + 3} + \dots + r_1)$$

$$= \frac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1),$$

$$\Sigma a'^2 = c^2[(r_2 + 1)^2 + (r_2 + 2)^2 + (r_2 + 3)^2 + \dots + r_1^2]$$

$$= \frac{1}{8}c^2[r_1(r_1 + 1)(2r_1 + 1) - r_2(r_2 + 1)(2r_2 + 1)],$$

$$\Sigma a'^3 = c^3 [(r_2 + 1)^3 + (r_2 + 2)^3 + (r_2 + 3)^3 + \dots + r_1^3]$$
  
=  $\frac{1}{4} c^3 [r_1^2 (r_1 + 1)^2 - r_2^2 (r_2 + 1)^2],$ 

$$\therefore M_{1} = \frac{-W}{l^{2}} \{ \frac{1}{2}c(r_{1} - r_{2})(r_{1} + r_{2} + 1)l^{2} - \frac{1}{3}c^{2}[r_{1}(r_{1} + 1)(2r_{1} + 1) - r_{2}(r_{2} + 1)(2r_{2} + 1)]l + \frac{1}{2}c^{3}[r_{1}^{2}(r_{1} + 1)^{2} - r_{2}^{2}(r_{2} + 1)^{2}] \},$$
 (283)

$$M_{2} = \frac{-W}{l^{2}} \{ \frac{1}{8} c^{2} [r_{1}(r_{1}+1)(2r_{1}+1) - r_{2}(r_{2}+1)(2r_{2}+1)] l - \frac{1}{4} c^{3} [r_{1}^{2}(r_{1}+1)^{2} - r_{2}^{2}(r_{2}+1)^{2}] \}, \quad (284)$$

which are the end moments due  $r_1 - r_2$  equal weights, W; both ends of beams fixed horizontally.

The greatest deflection due  $r_1 - r_2$  equal weights, W, placed at equal consecutive intervals anywhere along the beam, may be found by the following method:—

If in equation (281) we provisionally make x = cr, we shall have  $y_r$ . Then, putting r + 1 for r in this value of  $y_r$ , we find  $y_{r+1}$ ; and therefore

$$\Delta y = y_{r+1} - y_r.$$

Now, by making  $\Delta y = 0$ , we obtain a value of r the integral part of which, not less than  $r_1$  nor greater than  $r_2$ , will be the value of r in (281) when y is a maximum. Then, differentiating (281), and putting  $\frac{dy}{dx} = 0$ , we find a value of x which renders y a maximum.

Although this solution is rigorous, it need not often be employed, since (281) gives the deflection at as many points as we please, and a close approximation to the greatest value of y may be found by a few trials. An example will be given.

For finding the points of contrary flexure, we have, from (61) and (93),

$$M_{x} = \frac{W}{l} [(r_{1} - r)l - \frac{1}{2}c(r_{1} - r)(r_{1} + r + 1)]x - \frac{M_{1} - M_{2}}{l}x + M_{1}, \qquad (285)$$

which is the moment due the  $r_1 - r$  equal weights at any point, x, between the (r + 1)<sup>th</sup> weight and the left end of the beam; x being not greater than c(r + 1).

Also, from (60) and (93),

$$M_x = \frac{W_i}{2l}\dot{c}(r-r_2)(r+r_2+1)(l-x) - \frac{M_1-M_2}{l}x + M_1, \quad (286)$$

which is the moment due the  $r - r_2$  equal weights at any point, x, between the  $r^{th}$  weight and the right end of the beam; x not being less than cr.

If we now add equations (285) and (286), representing the three resulting moments still by the symbols  $M_x$ ,  $M_1$ ,  $M_2$ , we shall have

$$M_{x} = \left\{ \frac{W}{l} \left[ (r_{1} - r)l - \frac{1}{2}c(r_{1} - r_{2})(r_{1} + r_{2} + 1) \right] - \frac{M_{1} - M_{2}}{l} \right\} x + \frac{Wc}{2} (r^{2} - r_{2}^{2} + r - r_{2}) + M_{1}, \quad (287)$$

which is the moment due all the  $r_1 - r_2$  equal weights at any point, x, between the  $r^{th}$  and the  $(r + 1)^{th}$  weights; x being not less than cr, nor greater than c(r + 1), between  $r_2$  and  $r_3$ .

Now, at the points of contrary flexure, we have, in (287),  $M_x = 0$ ; and therefore

$$x = \frac{M_1 l + \frac{1}{2} W c l (r - r_2) (r + r_2 + 1)}{M_1 - M_2 - W \lceil (r_1 - r) l - \frac{1}{2} c (r_1 - r_2) (r_1 + r_2 + 1) \rceil}.$$
 (288)

But in this expression for x, whose value lies somewhere between cr and c(r + 1), we cannot tell what to call r. Let us,

therefore, in (288), put cr in the place of x, and determine the values of r, which must be integers; we can then find x, since the other quantities in (288) are given.

By putting cr for x in (288) we obtain

$$r = -s \pm \sqrt{\frac{2M_1}{cW} - r_2^2 - r_2 + s^2}, \qquad (289)$$

where

$$s = \frac{M_1 - M_2}{Wl} - r_1 - \frac{1}{2} + \frac{c}{2l}(r_1 - r_2)(r_1 + r_2 + 1).$$

If (289) gives values of r not integral, the decimals must be rejected, and the integers retained. Equation (289) will give r an integer only when there happens to be a point of contrary flexure at the r<sup>th</sup> interval; that is, where x really equals cr.

Having  $r_2$  equal intervals, c, without weights at the left end of the beam fixed horizontally at both ends, succeeded by  $r_1 - r_2$  equal weights, we have found the corresponding end moments in equations (283) and (284).

By making  $r_2 = 0$  in those equations, there results

$$M_{1} = \frac{Wcr_{1}(r_{1}+1)}{12l^{2}} [4c(2r_{1}+1)l - 6l^{2} - 3c^{2}r_{1}(r_{1}+1)], \quad (290)$$

$$M_2 = \frac{Wc^2r_1(r_1+1)}{12l^2} [3cr_1(r_1+1)-2(2r_1+1)l], \qquad (291)$$

which are the end moments due  $r_1$  equal weights, W, placed at equal intervals, c, along the beam fixed horizontally at both ends; the first weight being at the distance c from the left end.

83. Beam fixed Horizontally at the Right End, and simply supported at the Left, uniformly loaded for a Part or All of its Length with Equal Weights, W, at Equal Intervals, c. — If the first weight is  $r_2 + 1$  intervals from the free end, and if there are  $r - r_2$  equal weights, then the deflection at

any point, x, between the  $r^{th}$  interval and the horizontally fixed end of the beam is given by (271), provided we put therein

For a',

$$\frac{1}{2}c(r-r_2)(r+r_2+1).$$

For  $a^{\prime 3}$ ,

$$\frac{1}{4}c^3[r^2(r+1)^2-r_2^2(r_2+1)^2].$$

But if the first weight is at the distance c(r + 1) from the free end, and if there are  $r_1 - r$  equal weights at equal intervals, c, beyond, then the deflection at any point, x, between the (r + 1)<sup>th</sup> weight and the free end of the beam is given by equation (270) if there we substitute

For  $a'^{\circ}$ ,

$$r_{\tau}-r_{\cdot}$$

For  $\alpha'$ .

$$\frac{1}{4}c(r_1-r)(r_1+r+1).$$

For  $a^{\prime 2}$ ,

$$\frac{1}{8}c^2[r_1(r_1+1)(2r_1+1)-r(r+1)(2r+1)].$$

For  $\alpha'^3$ ,

$$\frac{1}{4}c^3[r_1^2(r_1+1)^2-r^2(r+1)^2].$$

If, then, we add the two deflections thus derived from (271) and (270), we shall have

$$y = \frac{W}{12Ell^{3}} \left\{ \left\{ \frac{3}{2}c(r_{1} - r_{2})(r_{1} + r_{2} + 1)l^{2} - \frac{1}{4}c^{3}[r_{1}^{2}(r_{1} + 1)^{2} - r_{2}^{2}(r_{2} + 1)^{2}] - 2(r_{1} - r)l^{3} \right\} x^{3} - 3c(r - r_{2})(r + r_{2} + 1)l^{3}x^{2} + \left\{ \frac{3}{4}c^{3}[r_{1}^{2}(r_{1} + 1)^{2} - (r_{2} + 1)^{2}r_{2}^{2}]l^{2} + \frac{3}{2}c(r_{1} - r_{2})(r_{1} + r_{2} + 1)l^{4} - c^{2}[r_{1}(r_{1} + 1)(2r_{1} + 1) - (r + 1)(2r + 1)r]l^{3} \right\} x - \frac{1}{2}c^{3}[r^{2}(r + 1)^{2} - (r_{2} + 1)^{2}r_{2}^{2}]l^{3} \right\}, \quad (292)$$

which is the deflection due all the  $r_1 - r_2$  equal weights at any point, x, between the  $r^{th}$  and the  $(r + 1)^{th}$  points of division; x being not less than cr nor greater than c(r + 1), but r from  $r_2$  to  $r_3$ .

In this case, where  $M_1 = 0$ ,  $M_2$  is derived from (268), which reduces to  $M_2 = \frac{W}{2l^2}(a'^3 - a'l^2)$ .

For a put

$$\frac{1}{2}c(r_1-r_2)(r_1+r_2+1).$$

For  $a'^3$  put

$$\frac{1}{4}c^3[r_1^2(r_1+1)^2-r_2^2(r_2+1)^2].$$

We then have

$$M_{2} = \frac{W}{2l^{2}} \left\{ \frac{1}{4} c^{3} \left[ r_{1}^{2} (r_{1} + 1)^{2} - r_{2}^{2} (r_{2} + 1)^{2} \right] - \frac{1}{2} c (r_{1} - r_{2}) (r_{1} + r_{2} + 1) l^{2} \right\}, \quad (293)$$

which is the end moment due  $r_r - r_2$  equal weights, W, uniformly distributed at equal intervals, c, on any part of the beam fixed horizontally at one end and simply supported at the other;  $r_1$  and  $r_2$  to be counted from the free end.

84. Deflection, End Moments, and Points of Contrary Flexure, due a Partial Uniform Load continuously distributed, when Both Ends of the Beam are fixed Horizontally.—We might proceed in this case as in article 74, using equations (53), (187), and (93); but, as the process is tedious, we employ the following method instead, utilizing results already obtained.

Let n denote, as heretofore, the whole number of intervals, each equal to  $(l \div n)$ . Let  $r_2$  denote a certain part of n, which we will call  $\frac{a}{l}n$ ; let  $r_1 = \frac{a+b}{l}n$ , where neither a nor a+b can exceed l.

$$c=\frac{l}{n}$$

Now, for a uniform continuous load we must have in the values of  $M_1$  and  $M_2$ , equations (290) and (291), n,  $r_2$ , and  $r_1$  infinite, and W infinitesimal; so that we must put nW = wl if wl = the weight per unit of the length.

Making these substitutions in (290) and (291), they become

$$M_{1} = \frac{w'}{12l^{2}} \{8[(a+b)^{3} - a^{3}]l - 6[(a+b)^{2} - a^{2}]l^{2} - 3[(a+b)^{4} - a^{4}]\}, \quad (294)$$

$$M_2 = \frac{w'}{12l^2} \{3[(a+b)^4 - a^4] - 4[(a+b)^3 - a^3]l\}, \qquad (295)$$

which are the end moments due the uniform continuous load w' per unit on the length b, measured to the right from a point at the distance a from the left end of the beam fixed horizontally at both extremities.

Now, if a = 0 (that is, if the continuous uniform load begins at the left end, and extends over the length b), equations (294) and (295) reduce to

$$M_1 = \frac{w'b^2}{12l^2}(8bl - 3b^2 - 6l^2), \qquad (296)$$

$$M_2 = \frac{w'b^2}{12l^2}(3b^2 - 4bl), \qquad (297)$$

which are the end moments due the continuous uniform load w' per unit, on the length b, measured from the left end.

It may be noted here, that if in (296) and (297), while a = 0, we suppose b = l, these values of  $M_1$  and  $M_2$  become each equal to  $-\frac{1}{12}w'l^2$ , which accords with equation (241) for the fully loaded beam.

Let us now put the values of  $M_1$  and  $M_2$ , as given by (294) and (295), into equation (236); we shall then have

$$y = \frac{w'}{24EIl^3} \Big\{ \{4[(a+b)^3 - a^3]l - 2[(a+b)^2 - a^2]l^2 - 2[(a+b)^4 - a^4]\} (x^3 - l^2x) - \{8[(a+b)^3 - a^3]l^2 - 6[(a+b)^2 - a^2]l^3 - 3[(a+b)^4 - a^4]l\} (x^2 - lx) \Big\},$$
 (298)

which is that part of the deflection due to the influence of the end moments, the beam horizontally fixed at both ends being loaded with w per unit for any part, b, of the beam's length, l; x varying from o to l.

If x be now restricted so as not to exceed a, and we add y in (298) to y in (225), the sum will be the deflection due w/b at any point between the origin and the beginning of the partial continuous uniform load w/b.

If again we limit x between the values a and a + b, and add the values of y in equations (298) and (226), the sum will be the deflection due in wb at any point of the loaded portion b.

Finally, by making x not less than a + b in (298), and adding that equation to (227), the sum of the second members will be the deflection due wb at any point between the right-hand end of the beam and the load wb.

It is evident, that, by assigning the proper values to a and b, we may place the load anywhere upon the beam, and give it any magnitude not exceeding w'l. Also, we may put many partial uniform continuous loads,  $w_1b_1$ ,  $w_2b_2$ ,  $w_3b_3$ , etc., upon the beam, by so choosing the values of  $a_1$ ,  $a_2$ ,  $a_3$ , etc.,  $b_1$ ,  $b_2$ ,  $b_3$ , etc., that the partial loads shall take desired positions, whether they are required to be equal to each other, or to overlap, or to have intervals between them.

But it is not necessary to formulate the deflection for such totals here.

It remains to find the points of contrary flexure for partial continuous uniform loads, w/b, when the beam is fixed horizontally at both ends.

If there is a point of contrary flexure between the left end of the beam and the beginning of the partial load (that is, within the length a), we use equations (53) and (93), giving

$$M_{x} = w'b\frac{l-a-\frac{1}{2}b}{l}x - \frac{M_{1}-M_{2}}{l}x + M_{1}.$$
If  $M_{x} = 0$ ,
$$x = \frac{M_{1}l}{M_{1}-M_{2}-w'b(l-a-\frac{1}{2}b)},$$
 (299)

where the values of  $M_1$  and  $M_2$  are to be taken from (294) and (295), and x cannot be greater than a. Should (299) yield a value of x either negative or greater than a, there is no point of contrary flexure in the part a.

For the loaded part of the beam b, we have equations (55) and (93), giving

$$M_{x} = w'b^{\frac{l-a-\frac{1}{2}b}{l}}x - \frac{1}{2}w'(x-a)^{2} - \frac{M_{1}-M_{2}}{l}x + M_{1} = 0,$$

$$\therefore x = 8 \pm \sqrt{\frac{2M_{1}}{w'} - a^{2} + 8^{2}}, \qquad (300)$$

where 
$$\epsilon = \frac{b(l-a-\frac{1}{2}b)}{l} - \frac{M_1-M_2}{w'l} + a$$
.

 $M_1$  and  $M_2$  are given by (294) and (295).

When, in (300), either value of x is less than a or greater than a + b, it must be rejected; and when both values of x are in this condition, there is no point of contrary flexure in the loaded part b.

In finding the point of contrary flexure between the right end of the beam and the load w'b, we employ equations (57) and (93); taking, as before, the values of  $M_1$  and  $M_2$  from (294) and (295). Thus,

$$M_{x} = w'b(a + \frac{1}{2}b)\frac{l - x}{l} - \frac{M_{1} - M_{2}}{l}x + M_{1} = 0,$$

$$\therefore x = \frac{M_{1}l + w'b(a + \frac{1}{2}b)l}{M_{1} - M_{2} + w'b(a + \frac{1}{2}b)}.$$
(301)

Equations (299) and (301) show that there can be but one point of contrary flexure between either end of the beam and the adjacent end of the load, while (300) indicates that there may be two such points within the length b covered by the uniform load w/b.

85. Partial or Full Continuous Uniform Load, w/b, on any Portion of a Beam fixed Horizontally at the Right End, but simply Supported at the Left. — Proceeding as in article 84, we make  $c = \frac{l}{n}$ ,  $r = \frac{a}{l}n$ ,  $r_1 = \frac{a+b}{l}n$ , and substitute these values in (293), which, when n is infinite, and W infinitesimal and  $=\frac{w/l}{n}$ , becomes

$$M_2 = \frac{w'}{8l^2} \{ (a+b)^4 - a^4 - 2l^2 [(a+b)^2 - a^2] \}, \quad (302)$$

which is the moment at the fixed end due the uniform continuous load, w/b, anywhere on the beam. Here, if a = 0, and b = l, the beam is fully covered by the load, and  $M_2 = -\frac{1}{8}w/l^2$ , in agreement with equation (246).

If in (302) a = 0, we have as the moment at the fixed end, when the partial load w/b, begins at the free end,

$$M_2 = \frac{w'}{8l^2}(b^4 - 2b^2l^2). \tag{303}$$

Substituting the value of  $M_2$  as given by (302), in equation (237), we obtain

$$y = \frac{w'}{48Ell^3} \Big\{ \{ (a+b)^4 - a^4 - 2l^2 [(a+b)^2 - a^2] \} (l^2x - x^3) \Big\}, \quad (304)$$

which is the deflection due the end moment  $M_2$  when  $M_1 = 0$ , and the load is w/b in any position; x varying from 0 to l.

If, as in article 84, x be now limited so as not to exceed a, and we add y in (304) to y in (225), the algebraic sum will be the deflection due w/b at any point, x, between the free end of the beam and the beginning of the load w/b.

If, again, x be limited between the values a and a + b, and we add algebraically the values of y in equations (304) and (226), the result will be the deflection due w/b at any point, x, of the loaded portion b.

Also, by making x not less than a + b in (304), and adding that equation to (227), the sum of the second members will be the deflection due wb at any point, x, between the right or fixed end of the beam and the load wb; x measured, as usual, from the free end of the beam.

The point of contrary flexure for the beam fixed horizontally at one end and simply supported at the other, which is taken as the origin, is found for a partial continuous uniform load, w'b, by means of equations (299), (300), and (301) for their respective cases, by putting  $M_1 = 0$ , and taking  $M_2$  from (302).

86. Examples illustrating Articles 82, 83. — For the sake of comparing the deflection of the same beam when one or both its ends are fixed, with its deflection when both ends are simply supported, we further consider the 15-inch rolled. wrought-iron I-beam of 30 feet clear span, whose moment of inertia I = 691, and whose modulus of elasticity E = 24,000,000, as given in article 75.

1st, Take 3 weights, of 4,500 pounds each, placed at intervals of 60 inches, beginning at the left end of the beam fixed horizontally at both ends; then the deflection at the centre is given by (278) if we put W = 4,500 pounds, l = 360 inches,  $c = \frac{1}{6}l = 60$ , r = 3, and  $x = \frac{1}{2}l$ ; EI being 16,584,000,000. Thus,

$$y = \frac{4500 \times 360^{3}}{6 \times 165840000000} \left\{ \left( \frac{1}{2} \times \frac{1}{36} \times 84 - \frac{1}{2} \times \frac{1}{216} \times 144 \right) \frac{1}{8} + \left( \frac{3}{4} \times \frac{144}{216} - \frac{84}{36} \right) \frac{1}{4} + \frac{1}{2} \times \frac{84}{36} \times \frac{1}{2} - \frac{1}{4} \times \frac{144}{216} \right\} = 0.13187 \text{ inch.}$$

2d, If 2 other equal weights, 4,500 pounds each, be added at the same interval of 60 inches, so as to cover the beam with concentrated loads, the central deflection due these last 2 is, by (280), where r = 3, and  $r_1 = 5$ , or by (278), making r = 2,

$$y = \frac{4500 \times 360^{3}}{6 \times 165840000000} \left\{ \left[ \frac{1}{2} \times \frac{1}{36} (330 - 84) - 2 - \frac{1}{2} \times \frac{1}{216} \times 756 \right] \frac{1}{8} + \left( \frac{3}{4} \times \frac{756}{216} - \frac{246}{36} + \frac{3 \times 18}{12} \right) \frac{1}{4} \right\} = 0.06594 \text{ inch.}$$

3d, For the 5 equal weights now on this beam, (281) gives the deflection

$$y = \frac{4500 \times 360^{3}}{6 \times 16584000000} \left\{ \left( \frac{1}{2} \times \frac{1}{36} \times 330 - 2 - \frac{900}{432} \right) \frac{1}{8} + \left( \frac{3}{4} \times \frac{900}{216} - \frac{330}{36} + \frac{54}{12} \right) \frac{1}{4} + \frac{1}{2} \times \frac{1}{36} \times \frac{84}{2} - \frac{1}{4} \times \frac{144}{216} \right\}$$

$$= 0.19781 \text{ inch};$$

or, (282) gives the same much more simply.

This value is, as it should be, the sum of the two deflections last found.

4th, Suppose the fifth weight removed from is the deflection at the fourth weight? U r = 4,  $c = \frac{1}{6}l = 60$  inches, W = 4.500 pounds

$$y = \frac{4500 \times 360^3 \times 16 \times 5}{24 \times 16584000000 \times 216} (13 - 4)$$

$$+ \frac{16 \times 33}{36} - \frac{2 \times 64 \times 5}{216}$$

5th, These 4 equal weights remaining on the deflection at the third weight, or centre? In equation (281), put  $x = \frac{1}{2}l = cr$ ,  $r = \frac{1}{2}$ 

$$y = \frac{4500 \times 360^{3}}{6 \times 16584000000} \left\{ \left( \frac{180}{7^{2}} - 1 - \frac{400}{43^{2}} \right) \frac{1}{8} + \left( \frac{3 \times 400}{4 \times 216} - \frac{180}{36} + \frac{24}{12} \right) \frac{1}{4} + \frac{84}{7^{2}} - \frac{144}{4 \times 216} \right\}$$

6th, The same 4 weights remaining, what at the second weight?

Use (281), calling  $r_1 = 4$ , r = 2, x = rc =

$$\therefore y = \frac{4500 \times 360^3}{6 \times 16584000000} \left\{ \left( \frac{180}{7^2} - 2 - \frac{400}{43^2} \right) \frac{1}{27} + \left( \frac{3}{4} \times \frac{400}{216} - \frac{180}{36} + \frac{4^2}{12} \right) \frac{1}{9} + \frac{30}{216} - \frac{1}{2} \right\}$$

7th, What are the end moments due these same position as above?

Use (283) and (284), making r = 4, c = 4,500;

$$M_{1} = \frac{4500 \times 360}{12} \times \frac{20}{6} \left( 4 \times \frac{1}{6} \times 9 - 6 - \frac{1}{6} \times 9 - 6 - \frac{1}{6} \times 9 - 6 \right)$$

$$= -7$$

$$\times 9 = -6$$

8th, When all the 5 weights are on the beam uniformly distributed as above, r = 5,  $c = \frac{1}{6}l = 60$ , W = 4,500. Then, by (283) and (284),

$$M_1 = \frac{4500 \times 360}{12} \times \frac{30}{6} \left( \frac{44}{6} - 6 - \frac{90}{36} \right) = -787500$$
 inch-pounds.

$$M_2 = \frac{4500 \times 360}{12} \times \frac{30}{36} \left( \frac{3}{6} \times 30 - 2 \times 11 \right) = -787500$$
 inch-pounds.

9th, The 4 equal weights of 4,500 pounds still occupying the first 4 intervals on this beam, where are the points of contrary flexure? Here we have  $M_1 = -750,000$ ,  $M_2 = -600,000$ , W = 4,500,  $c = \frac{1}{6}l = 60$ ,  $r_1 = 4$ ,  $r_2 = 0$ .

These values put in (289) give

$$r = 4.6595$$
 or 1.1923.

We have then, rejecting the decimals, r = 1 or 4. Hence (288) becomes

$$x = \frac{-750000 + \frac{1}{2} \times 4500 \times \frac{360}{8} \times 2}{-\frac{150000}{360} - 4500(3 - \frac{1}{2} \times \frac{1}{8} \times 4 \times 5)} = 74.806 \text{ for } r = 1,$$

$$x = \frac{-750000 + \frac{1}{2} \times 4500 \times \frac{360}{8} \times 20}{-\frac{150000}{360} - 4500(0 - \frac{1}{2} \times \frac{1}{6} \times 4 \times 5)} = 275.294 \text{ for } r = 4.$$

10th, If these 4 weights occupy the last 4 intervals, leaving the first vacant, we shall have  $M_1 = -600,000$ ,  $M_2 = -750,000$ ,  $r_1 = 5$ ,  $r_2 = 1$ ,  $c = \frac{1}{6}l = 60$ , W = 4,500; so that, from (289), we find r = 4.80372 or 1.34442, that is, 4 or 1.

These values placed in (288) give

$$x = 84.706 \text{ for } r = 1,$$

x = 285.194 for r = 4,

nich accords with example oth, since 360 — 84.70

which accords with example 9th, since 360 - 84.706 = 275.294, and 360 - 285.194 = 74.806.

11th, When all 5 weights are on the beam at equal intervals,  $M_1 = M_2 = -787,500$  by example 8th. Also,  $c = \frac{1}{6}l = 60$ ,  $r_1 = 5$ ,  $r_2 = 0$ , W = 4,500. From (289), we find that r must be 1 or 4; and therefore (288) gives, as the points of contrary flexure,

$$x = 76\frac{1}{3}$$
 for  $r = 1$ ,  
 $x = 283\frac{1}{3}$  for  $r = 4$ .

The sum of these values of x is 360, as it should be, since the load is symmetrical.

12th, Let there be on this beam weights at the end of the second and third intervals, and find the end moments and points of contrary flexure. We now have W=4,500,  $c=\frac{1}{6}l=60$  inches,  $r_1=3$ ,  $r_2=1$ ; so that (290) and (291) become

$$M_{1} = \frac{-4500 \times 360}{12} \left\{ \frac{6}{6} \times 2 \times 5 - 4 \times \frac{1}{36} (3 \times 4 \times 7 - 1 \times 2 \times 3) + \frac{3}{216} (144 - 4) \right\} = -442500,$$

$$M_{2} = \frac{-4500 \times 360}{12} \left\{ \frac{2}{36} (84 - 6) - \frac{3}{216} (144 - 4) \right\} = -322500.$$

Using these moments in (289), we find r = 1.2460 or 4.2354; we use r = 1 or 4. Therefore, from (288), x = 79.254 or 251.053, which are the points of contrary flexure sought.

13th, When these 2 equal weights are at the second and third points of division, as in the twelfth example, what is the maximum deflection of the beam, and at what point does it occur?

Using (281), where now  $W = 4.5\infty$ ,  $c = 60 = \frac{1}{6}l$ ,  $r_1 = 3$ ,  $r_2 = 1$ , and provisionally putting x = cr, we find  $y_r$ ; then, putting r + 1 for r in the value of  $y_r$ , we find  $y_{r+1}$ ; and then, making  $\Delta y = y_{r+1} - y_r = 0$ , we obtain

$$r^3 - 6.722r^2 + 9.111r + 2.676 = 0$$

from which we easily see, as was suspected, that a positive value of r lies between 2 and 3 for a maximum y.

Making, therefore, r=2 in equation (281), and differentiating with respect to x, then putting  $\frac{dy}{dx}=0$ , we find

$$x = 0.47391$$

which, substituted in (281), r being 2, gives

$$y = 0.11553$$
 inch, a maximum.  
At centre,  $y = 0.11478$  inch, at second weight.  
At  $\frac{1}{3}$ ,  $y = 0.09514$  inch, at first weight.

87. When the uniform discontinuous load is applied at equal consecutive intervals, the first weight being placed at no integral number of times the common interval from the left end of the beam, we may proceed in finding the deflection, end moments, and points of contrary flexure as in article 20, where r,  $r_1$ , and  $r_2$  need not be integral, but where the differences,  $r_1 - r_2$ ,  $r_1 - r_2$ ,  $r_1 - r$ , each denoting a number of weights, must be integral. In this way the deflection formulæ already established in this chapter for full intervals, r,  $r_1$ ,  $r_2$ , being whole numbers, also apply to the case now under consideration, where r, r, and r, have the same fractional part, except that, when r is negative, its value is less, by unity, than the common decimal part of r and r, as before shown.

Using equation (292), where now  $r_2 = -\frac{1}{2}$ ,  $r_1 = 5\frac{1}{2}$ ,  $r = 2\frac{1}{2}$ , and  $x = \frac{1}{2}l = 180$ , we find central deflection

$$y = \frac{4500 \times 360^{3}}{12 \times 16584000000} \left\{ \frac{1}{8} \left( \frac{3}{2} \cdot \frac{1}{6} \cdot 6^{2} \right) - \frac{1}{4} \cdot \frac{1}{216} \left[ \left( \frac{11}{2} \right)^{2} \left( \frac{13}{2} \right)^{2} - \frac{1}{16} \right] - 2 \times 3 \right\} - \frac{1}{4} \left( \frac{3^{3}}{6} \right) + \frac{1}{2} (4.4375 + 9 - 10.4583) - 0.1771 \right\} = 0.3993 \text{ inch.}$$

And the greatest deflection due this full load on the beam fixed horizontally at the right-hand end is found by putting x = cr provisionally in (292), and making  $y_{r+1} - y_r = 0 = \Delta y$ . This equation indicates a value of r between  $\frac{3}{2}$  and  $\frac{5}{2}$ .

Calling  $r = \frac{3}{2}$  in (292), and putting  $\frac{dy}{dx} = 0$ , we find x = 0.42077l = 151.477 inches, which is greater than  $c(r + 1) = 60(\frac{3}{2} + 1) = 150$  inches, an inadmissible result. Hence we see that the approximate equation  $\Delta y = 0$  gave r too small. Now, calling  $r = \frac{5}{2}$  in (292), and making  $\frac{dy}{dx} = 0$ , we find x = 0.417404l = 150.265 inches, which is between cr and c(r + 1), as it should be.

With  $r = \frac{5}{2}$ , and x = 0.417404l, (292) gives greatest deflection y = 0.41463 inch; while at the centre it was 0.3993 inch. The end moment in this case removes the point of greatest deflection 180 - 150.265 = 29.735 inches from the centre.

The end moment due this load is given by (293), where  $r_1 = \frac{11}{2}$ ,  $r_2 = -\frac{1}{2}$ ,  $c = \frac{1}{6}l = 60$ , and W = 4,500 pounds; and it is, in inch-pounds,

$$M_2 = \frac{4500 \times 360}{2} \left\{ \frac{I}{4} \times \frac{I}{216} \left[ \left( \frac{II}{2} \right)^2 \left( \frac{I3}{2} \right)^2 - \frac{I}{16} \right] - \frac{I}{2} \cdot \frac{I}{6} \cdot 6^2 \right\}$$
  
= -1231875.

The point of contrary flexure is found by adding equations (62) and (93), and equating the sum of the second members to zero.

Thus, since  $M_1 = 0$ , we have

$$M_{x} = \left\{ \frac{W}{2l} \left[ 2(r_{1} - r)l - c(r_{1} - r_{2})(r_{1} + r_{2} + 1) \right] + \frac{M_{2}}{l} \right\} x + \frac{1}{2} Wc(r - r_{2})(r + r_{2} + 1) = 0, \quad (305)$$

$$\therefore x = \frac{-c(r-r_2)(r+r_2+1)l}{2l(r_1-r)-c(r_1-r_2)(r_1+r_2+1)+\frac{2M_2}{W}}.$$
 (306)

Making x = rc provisionally in (306), we find r = 4.5353. Calling  $r = \frac{9}{2}$ , and  $M_2 = -1,231,875$ , (306) yields the point of contrary flexure

$$x = 0.7547171$$

which is between rc and (r + 1)c (that is, between 0.75l and  $\frac{11}{2}l$ ), though very close to the former.

If both ends of this beam are free under this load of 6 equal weights, we find by (232), at the point x = 0.417404l, y = 0.9676 inch.

And, by (237), the deflection due  $M_2 = 1,231,875$  is y = -0.5530 inch, which added to 0.9676 gives y = 0.4146 inch, as found by (292) above.

88. Continuous Uniform Load, w/b, on Beam fixed Horizontally at Both Ends.—Take the examples of article 75, and apply to the deflections there found the effects of the end moments as given by equation (298).

1st, In the first example of article 75, for beam with free ends, the deflection, when  $x = b = \frac{1}{3}l = 120$  inches, and a = 0, was found to be y = 0.23444 inch.

Now, by (298), the effect of end moments on the deflection in this case is

$$y = \frac{75 \times 360^4}{24 \times 16584000000} \left\{ \left( \frac{4}{27} - \frac{2}{9} - \frac{2}{81} \right) \left( \frac{1}{27} - \frac{1}{3} \right) - \left( \frac{8}{27} - \frac{6}{9} - \frac{3}{81} \right) \left( \frac{1}{9} - \frac{1}{3} \right) \right\} = -0.19392 \text{ inch.}$$

Therefore the deflection sought is

$$y = 0.23444 - 0.19392 = 0.04052$$
 inch;

the left third of the 15-inch I-beam bearing 75 pounds to the inch, both ends being fixed horizontally.

2d, Again, in the second example of article 75 the central deflection = 0.24421 inch, under the same conditions. If, now, in (298) we make a = 0,  $b = \frac{1}{8}l = 120$  inches,  $x = \frac{1}{2}l$ , we get the effect of end moments on deflection

$$y = \frac{75 \times 360^4}{24 \times 16584000000} \left\{ -\frac{8}{81} \left( \frac{1}{8} - \frac{1}{2} \right) + \frac{33}{81} \left( \frac{1}{4} - \frac{1}{2} \right) \right\} = -0.20514 \text{ inch.}$$

Therefore the required deflection is

$$y = 0.24421 - 0.20514 = 0.03907$$
 inch

at the centre of the beam fixed horizontally at both ends.

3d, Applying the value of y in (298) to the deflection found in the third example of article 75, where a = 0,  $b = \frac{1}{8}l$ ,  $x = \frac{2}{8}l$ , we find, for the beam with fixed ends,

$$y = 0.19176 - 0.17077 = 0.02099$$
 inch.

4th, The greatest deflection due 75 pounds per inch on the left third of this I-beam fixed at both ends, is found by adding equations (229) and (298), and in the resulting equation making  $\frac{dy}{dx} = 0$  when a = 0, and  $b = \frac{1}{8}l$ .

This gives  $x = \frac{2}{6}l$ , whence the greatest deflection y = 0.042199 inch at  $\frac{2}{6}l$  from the left end of the beam, which is  $(\frac{2}{6} - \frac{2}{6})l = \frac{1}{15}l$  beyond the end of the load.

5th, The end moments for this load of 75 pounds per inch on the left third of this 15-inch I-beam 30 feet long, where I = 691, E = 24,000,000, a = 0,  $b = \frac{1}{8}l = 120$  inches, are given by equations (296) and (297), as follows:

$$M_1 = \frac{75 \times \frac{1}{9}l^4}{12l^2} \left(\frac{8}{3} - \frac{3}{9} - 6\right) = -330000$$
 inch-pounds,

$$M_2 = \frac{75 \times 360^2}{108} \left(\frac{3}{9} - \frac{4}{3}\right) = -80000$$
 inch-pounds.

6th, With these values of  $M_1$  and  $M_2$ , equation (300) gives the first point of contrary flexure,

$$x = 100.926 \pm 37.229 = 63.697$$
 inches,

since in (300) x cannot be greater than  $(a + b) = (0 + \frac{1}{8}l) =$  120 inches.

The second point of contrary flexure is derived from (301), where we find x = 260.69 inches.

The mode of procedure when only one end of the beam is fixed horizontally is so similar to that just exemplified for two fixed ends, that further examples seem to be unnecessary.

## Section 4.

Deflection of a Girder of Variable Cross-Section in Terms of the Constant Unit Strain upon the Extreme Fibres of the Section; that is, Deflection of a Beam of Uniform Strength. End Moments for Fixed Beams.

89. Economy in the construction of built beams or framed girders requires that the cross-sections of the various members, as well as that of the whole structure, should be proportioned to the greatest strains allowed upon the sections; and, when the dimensions of parts are so adjusted, it is clear that the unit strain of tension, compression, or bending will be constant throughout the girder.

The complete realization of this condition is, for obvious considerations, probably seldom attained; but it is a condition so nearly approximated in practice as to require examination here.

For this case we employ equation (186); viz.,

$$-E\frac{d^2y}{dx^2}=\frac{2B_1}{h},$$

which is independent of I, the moment of inertia of the cross-section, and in which  $B_i$  is constant for a given load, and equal to the mean of the unit strains upon the fibres at the upper and lower surfaces of the beam, and h = height of cross-section.

90. Deflection of Semi-Girder of Uniform Height, h, and Uniform Strength. — Using the notation of article 64, as illustrated by Fig. 8, and integrating (186), with the sign of  $E\frac{d^2y}{dx^2}$  positive for the semi-girder, first, with the condition that  $\frac{dy}{dx} = 0$  when x = 0,

$$\therefore E\frac{dy}{dx} = \frac{2B_1}{h}x;$$

secondly, y = 0 when x = 0,

$$\therefore Ey = \frac{B_{x}x^{2}}{h},$$

$$\therefore y = \frac{B_{x}x^{2}}{Eh},$$
(307)

which is the deflection at any point, x, of the semi-girder of uniform height and strength.

If 
$$x = l$$
,
$$D = \frac{B_1 l^2}{Eh},$$
(308).

which is the deflection at the free end of the semi-girder of uniform height and strength.

It may be observed that (307) is the equation of a parabola with its vertex at the origin of co-ordinates.

Example. — Take an open-webbed semi-girder of wroughtiron whose effective height, h, is 20 feet = 240 inches, length, l, = 50 feet = 600 inches; and suppose the allowed unit strain in the top chord is  $C_1 = 8,000$  pounds per square inch, and in the bottom chord  $T_1 = 10,000$  pounds per square inch. Then calling, as we may do without sensible error, the top and bottom chords extreme fibres of the cross-section, we have

$$B_{\rm r} = \frac{1}{2}(C_{\rm r} + T_{\rm r}) = 9000. \tag{309}$$

Take E = 25,000,000, then

Deflection at free end = 
$$D = \frac{9000 \times 600^2}{240 \times 25000000} = 0.54$$
 inch,

Deflection at centre = 
$$\frac{9000 \times 300^2}{240 \times 25000000}$$
 = 0.135 inch.

It should be remembered that the deflection of a framed girder due to its first full load is likely to be greater than that computed by these formulæ, by reason of the yielding of the joints and probable straightening of some of the parts in tension. It is customary, therefore, in computing the deflection of a girder under its first loads until the frame becomes "set," to take E ranging from 15,000,000 to 20,000,000 for wrought-iron, according to the accuracy of the joint fittings and general workmanship; afterwards the ordinary value of E may be used.

91. Deflection of the Semi-Girder of Uniform Strength but of Variable Height.—(a) Let the semi-girder be like either half of Fig. 64, 65, 67, 33, 34, 39, or 83; that is, let it slope uniformly from the fixed end, whose height we will call  $h_1$ , to the free end, whose height is  $h_2$ .

Then the height at any point, x, is

$$h = h_1 - \frac{h_1 - h_0}{l}x;$$

$$dh = \frac{h_0 - h_1}{l}dx,$$

$$dx^2 = \left(\frac{l}{h_0 - h_1}dh\right)^2.$$
(310)

and

Hence (186) becomes, for this semi-girder,

$$\frac{E(h_0 - h_1)^2}{2B_1l^2} \cdot \frac{d^2y}{dh^2} = \frac{1}{h}.$$

Integrating, with the condition that  $\frac{dy}{dh} = 0$  when  $h = h_{i}$ ,

$$\therefore \frac{E(h_0-h_1)^2}{2B_1l^2} \cdot \frac{dy}{dh} = \log_e \frac{h}{h_1},$$

where log. denotes the Napierian logarithm.

Again, y = 0 when  $h = h_{i}$ ,

$$\therefore \frac{E(h_0-h_1)^2}{2B_1l^2}y = \int_{h_1}^{h} \log_{\epsilon}\frac{h}{h_1} \cdot dh = h\left(\log_{\epsilon}\frac{h}{h_1}-1\right) + h_1,$$

$$\therefore y = \frac{2B_1l^2}{E(h_0 - h_1)^2} \Big\{ h_1 - h \Big( 2.302585 \log \frac{h_1}{h} + 1 \Big) \Big\}, \quad (311)$$

which is the deflection of the uniformly sloping semi-girder at any point where the height is h; log denoting the common logarithm, and the girder being of uniform strength.

Putting for h in (311) its value as taken from (310), we have y in terms of x; thus,

$$y = \frac{2B_1 l^2}{E(h_0 - h_1)^2} \left\{ h_1 - \left( h_1 - \frac{h_1 - h_0}{l} x \right) \left( 2.302585 \log \frac{h_1 l}{h_1 l + (h_0 - h_1) x} + 1 \right) \right\}, \quad (312)$$

which is the same as (311).

If the semi-girder of uniform slope and strength comes to a point at the free end, we have at that end  $h_0 = 0 = h$ ; and therefore (311) becomes

$$D = \frac{2B_1l^2}{Eh_1},$$
 (313)

which is twice the deflection given by (308) for semi-beam of the same length but of the uniform height  $h_{\rm r}$ .

When  $h_0 = h_1 = h$ , the value of y in (311) and (312) is indeterminate, but is given by (307).

EXAMPLE.. — Length of semi-girder l = 50 feet; height at fixed end = 20 feet, at free end 10 feet;  $B_1 = 9,000$ ; E = 25,000,000 pounds per square inch. What is the deflection at the free end?  $h_1 = 240$  inches,  $h_0 = h = 120$  inches, l = 600 inches.

By (311),

$$y = \frac{2 \times 9000 \times 600^2}{25000000 \times (-120)^2} [240 - 120(2.302585 \log 2 + 1)]$$
  
= 0.6628 inch.

(b) Semi-Girder with Either or Both Chords Parabolic. Open Frame. — First, take a case like the half of Fig. 63, supposing the top chord parabolic, and, as in all these cases, the members formed as for a semi-beam. Let l = length of semi-girder,  $h_1 = \text{its height at the fixed end}$ ,  $h_0 = \text{height at free end}$ , and h = variable height. Then, by equation (136), putting for the h in that equation  $h_1 - h_0$ , and adding  $h_0$  to the second member for our present case, we have, l also being put for  $\frac{1}{2}l$ ,

$$h = h_1 - \frac{h_1 - h_0}{l^2} x^2. (314)$$

This value of h placed in (186) gives, after reducing, and making  $m^2 = \frac{h_1}{h_1 - h_2}$ ,

$$\frac{(h_{\rm I}-h_{\rm o})E}{2B_{\rm I}l^2}\cdot\frac{d^2y}{dx^2}=\frac{1}{m^2l^2-x^2}.$$
 (315)

Integrating (315), first with the condition  $\frac{dy}{dx} = 0$  when x = 0,

$$\therefore \frac{m(h_1 - h_0)E}{B_1 l} \cdot \frac{dy}{dx} = \log_{\bullet} \frac{ml + x}{ml - x}, \qquad (316)$$

where log, means Napierian logarithm.

Integrating again, with the condition y = 0 when x = 0,

$$\therefore \frac{m(h_1 - h_0)E}{B_1 l} y = 2.302585 [(ml + x)\log(ml + x) + (ml - x)\log(ml - x) - 2ml\log ml],$$

$$y = \frac{2.302585B_1l}{m(h_1 - h_0)E} [(ml + x)\log(ml + x) + (ml - x)\log(ml - x) - 2ml\log ml], \quad (317)$$

where log denotes common logarithm, and y is the deflection at any point, x, of the semi-girder of uniform strength, and of the form of one-half of Fig. 63, when the top chord is parabolic.

Example. — Let  $B_1 = 9,000$ , E = 25,000,000, l = 600 inches,  $h_1 = 240$  inches,  $h_0 = 120$  inches.

$$m=\sqrt{\frac{h_1}{h_1-h_0}}=\sqrt{2}.$$

If, now, x = l, we have the deflection at the free end of the girder, from (317),

$$y = \frac{2.302585 \times 9000 \times 600}{25000000 \times 120\sqrt{2}} (203.9) = 0.59757 \text{ inch,}$$

which is, as it manifestly should be, less than the deflection just found by (311) for the semi-beam of equal length and depth of ends, but of uniform slope, and greater than the deflection of semi-beam of same length and uniform depth  $= h_1$ , found by equation (308).

If this girder comes to a point at the free end (that is, if it is the half of the parabolic bowstring), we have, in (317),  $h_0 = 0$ , m = 1;

$$\therefore y = \frac{2.302585B_1 l}{Eh_1} [(l+x)\log(l+x) + (l-x)\log(l-x) - 2l\log l], \quad (318)$$

which is the deflection at any point, x.

When, in (318), x = l, we have the deflection at the free end of the parabolic semi-bowstring; thus,

$$D = \frac{1.386295B_1l^2}{Eh_1},$$
 (319)

which, according to (313), is  $\frac{1.386295}{2}$  of the deflection at the free end of the semi-girder of same length and height at fixed end, but sloping uniformly to a point.

From the identity in the form of equations (136), (137), and (138), and from the manner in which (317), (318), and (319) have been derived from (136), it follows that the deflection of any parabolic semi-girder of uniform strength, whether the half-crescent, or the half double bowstring, may be found from (317), (318), and (319), provided we make  $h_1$  = the height of girder at fixed end, and  $h_0$  = its height at the free end.

(c) Semi-Girder with Circular Arc for Top Chord. Uniform Strength. — Let, as before,  $h_1$  = height at fixed end,  $h_0$  = height at free end, for a girder like the right half of Fig. 63, fixed at the vertical plane through the centre; the top chord being now supposed circular.

If R is the radius of the circle, the height of the semi-girder at any point, x, is given by equation (125),

$$h = h_1 - h_0 + h_0 + \sqrt{R^2 - x^2} - R,$$

$$h = h_1 - R + \sqrt{R^2 - x^2},$$

$$h = h_1 - R + R\cos\theta;$$
(320)

 $\theta$  being the arc between the point (x, y) of equation (125) and the fixed end of the girder.

Therefore (186) becomes

$$\frac{E}{2B_1}\frac{d^2y}{dx^2} = \frac{1}{h_1 - R + R\cos\theta}.$$
 (322)

But  $x = R \sin \theta$ ,

$$\therefore dx = R \cos \theta \, d\theta,$$

$$\therefore \frac{E}{2B} \frac{d^2y}{dx} = \frac{\cos\theta \, d\theta}{a + \cos\theta}, \tag{323}$$

if 
$$a = \frac{h_i - R}{R}$$
.

Integrating first with the condition that  $\frac{dy}{dx} = 0$  when  $\theta = 0$ , we have (for this first integration, see Price, "Infinitesimal Calculus," vol. ii. p. 85), after reducing, and putting  $a = \cos \alpha$ ,

$$\frac{E}{2B_1}\frac{dy}{dx} = \theta - \frac{a}{\sin\alpha}\log_{\epsilon}\frac{\cos\frac{\alpha - \theta}{2}}{\cos\frac{\alpha + \theta}{2}}$$
 (324)

where, as usual, log, means Napierian logarithm.

For the second integration, between the limits 0 and y, 0 and  $\theta$ , (324) takes the form

$$\frac{E}{2B_1R}dy = \theta\cos\theta\,d\theta - \frac{a}{\sin\alpha}\log_{\epsilon}\frac{\cos\frac{\alpha-\theta}{2}}{\cos\frac{\alpha+\theta}{2}}\cos\theta\,d\theta. \quad (325)$$

The first term is easily integrated thus:

$$\int_{0}^{\theta} \theta \cos \theta \, d\theta = \theta \sin \theta - \int_{0}^{\theta} \sin \theta \, d\theta$$
$$= \left[ \theta \sin \theta + \cos \theta \right]_{0}^{\theta}$$
$$= \theta \sin \theta + \cos \theta - 1.$$

Integrate the second term also by parts, according to the form

$$\int u \, dv = uv - \int v \, du. \tag{326}$$

Take 
$$u = \log_{\epsilon} \frac{\cos \frac{\alpha - \theta}{2}}{\cos \frac{\alpha + \theta}{2}} = \log_{\epsilon} \cos \frac{\alpha - \theta}{2} - \log_{\epsilon} \cos \frac{\alpha + \theta}{2}$$
.

$$dv = \cos\theta \, d\theta, \qquad \therefore \quad v = \sin\theta.$$

$$du = \frac{\frac{1}{2}\sin\frac{\alpha-\theta}{2}}{\cos\frac{\alpha-\theta}{2}}d\theta + \frac{\frac{1}{2}\sin\frac{\alpha+\theta}{2}}{\cos\frac{\alpha+\theta}{2}}d\theta = \frac{1}{2}\left(\tan\frac{\alpha-\theta}{2} + \tan\frac{\alpha+\theta}{2}\right)d\theta.$$

Therefore the second term of the second member of (325) becomes

$$-\frac{\alpha}{\sin\alpha}\left\{\sin\theta\log_{\alpha}\frac{\cos\frac{\alpha-\theta}{2}}{\cos\frac{\alpha+\theta}{2}}-\frac{1}{2}\int\left(\tan\frac{\alpha-\theta}{2}+\tan\frac{\alpha+\theta}{2}\right)\sin\thetad\theta\right\}.$$

But

$$\tan\frac{\alpha-\theta}{2}+\tan\frac{\alpha+\theta}{2}=\frac{2\sin\alpha}{\cos\alpha+\cos\theta}$$

and the second term reduces to

$$-\frac{a\sin\theta}{\sin\alpha}\log_{\epsilon}\frac{\cos\frac{\alpha-\theta}{2}}{\cos\frac{\alpha+\theta}{2}}+a\int\frac{\sin\theta\,d\theta}{\cos\alpha+\cos\theta}.$$

Now

$$a\int \frac{\sin\theta \,d\theta}{\cos\alpha + \cos\theta} = -a\int \frac{d(\cos\alpha + \cos\theta)}{\cos\alpha + \cos\theta} = -a\log_{\epsilon}(\cos\alpha + \cos\theta).$$

Whence, finally, the integral of (325) is

$$\frac{E}{2B_1R}y = \left\{\theta\sin\theta + \cos\theta - \frac{a\sin\theta}{\sin\alpha}\log_{\epsilon}\frac{\cos\frac{\alpha - \theta}{2}}{\cos\frac{\alpha + \theta}{2}} - a\log_{\epsilon}(\cos\alpha + \cos\theta)\right\}_{0}^{\theta},$$

$$\therefore y = \frac{2B_1R}{E}\left\{\theta\sin\theta + \cos\theta - 1\right\}$$

$$-\frac{a}{m_1}\left(\frac{\sin\theta}{\sin\alpha}\log\frac{\cos\frac{\alpha - \theta}{2}}{\cos\frac{\alpha + \theta}{2}} + \log\frac{a + \cos\theta}{a + 1}\right), (327)$$

where  $m_1 = 0.4342945$ , the modulus of common logarithms, log; and y is the deflection at any point,  $x = R \sin \theta$ , of the semi-girder having its top chord circular and bottom chord straight, like the truncated bowstring.

When  $h_0 = 0$  (that is, when the semi-girder is half of the common bowstring girder), the last term of (327) becomes infinite for  $a = -\cos\theta$ , which is the case if x = l = l length of semi-girder, and  $\sin\theta = l \div R$ .

But in this case  $\sin \theta = \sin \alpha$ ; and (327) is easily reduced to

$$y = \frac{2B_1R}{E} \left\{ \theta \sin \theta + \cos \theta - 1 + 2.302585a \log \frac{a+1}{2\left(\cos \frac{\alpha-\theta}{2}\right)^2} \right\}, (328)$$

which is the deflection at the free end of the circular semibow-string of uniform strength.

EXAMPLE 1. — Semi-bowstring. l = 600 inches,  $h_1 = 240$  inches,  $B_1 = 9,000$ , E = 25,000,000, wrought-iron.

:. 
$$R = \frac{l^2 + h_1^2}{2h_1} = 870$$
 inches = 72.5 feet,

$$\sin \theta = l + R = \frac{60}{87} = 0.689655, \quad \theta = 43^{\circ} 36' \text{ io''.15},$$

$$\cos \theta = \frac{R - h_1}{R} = \frac{63}{87} = 0.724138.$$
In arc, 
$$\theta = \frac{43.60282}{180}\pi = 0.761013,$$

$$a = \frac{h_1 - R}{R} = -\frac{63}{87} = -0.724138 = \cos \alpha = -\cos \theta,$$

$$\therefore \alpha = 180^{\circ} - 43^{\circ} 36' \text{ io''.15} = 136^{\circ} 23' 49''.85,$$

$$\frac{1}{2}(\alpha - \theta) = 46^{\circ} 23' 49''.85.$$

Therefore the deflection at the free end of this semi-bowstring of uniform strength is, by (328),

$$y = \frac{2 \times 9000 \times 870}{25000000} (0.524836 + 0.724138 - 1 + 1.145371)$$
  
= 0.71745 inch,

which is a little less than  $\frac{1.386295}{2} \times 1.08 = 0.7486$  inch = deflection at free end of parabolic semi-bowstring, by (319). And this should be so, since the top chord of the parabolic girder lies just below that of the circular bowstring of the same central height and same span.

EXAMPLE 2. — Semi-girder, truncated bowstring, circular. l = 600 inches,  $h_1 = 240$ ,  $h_0 = 120$ ,  $B_1 = 9,000$ , wrought-iron; E = 25,000,000;

$$\therefore R = \frac{l^2 + (h_1 - h_0)^2}{2(h_1 - h_0)} = 130 \text{ feet} = 1560 \text{ inches,}$$

Use equation (327).

$$\sin \theta = l + R = \frac{6}{18},$$

$$\theta = 22^{\circ} 37' \text{ II''}.5 = \frac{22.619861}{180} \pi = 0.39479 \text{ in arc.}$$

$$\cos \theta = 0.923077, \quad a = \cos \alpha = \frac{20 - 130}{130} = -\frac{11}{13};$$

$$\sin \alpha = 0.532939, \quad \alpha = 180^{\circ} - 32^{\circ} 12' 15''.3 = 147^{\circ} 47' 44''.7;$$

$$\frac{\alpha + \theta}{2} = 85^{\circ} 12' 28''.1, \quad \frac{\alpha - \theta}{2} = 62^{\circ} 35' 16''.6;$$

$$\therefore \quad y = \frac{2 \times 9000 \times 1560}{25000000} \text{ (0.151842} + 0.923077 - 1 + 0.455710)$$

$$= 0.596003 \text{ inch,}$$

which is the deflection at the free end, and is, as was to be expected, a little less than that found by (317) for the parabolic semi-girder of the same length and end heights.

- 92. Equations (327) and (328) apply also to the double circular bowstring, truncated or otherwise, provided the radii of the two curves are the same. But when these radii are different, we may, without sensible error, employ the equations (317), (318), and (319), deduced for the deflection of the parabolic semi-girder of uniform strength, and applicable to all the cases, including the crescent and the double bow; the computed deflection being always a little greater than that due the circular semi-girder of the same end heights and span.
- 93. Deflection of the Girder of Uniform Strength supported at Both Ends, either Fixed or Free, and the Height of the Girder being either Uniform or Variable. Since the deflection of a girder may be defined as the difference of level between the position of any one of its points before bending and the position of the same point after bending, under

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$$\frac{1}{3}(\alpha + \theta) = 86^{\circ} 39' 31''.15, \quad \alpha = 180^{\circ} - 18^{\circ} 6' 14'' = 161^{\circ} 53' 46''.$$

$$\frac{1}{3}(\alpha - \theta) = 75^{\circ} 14' 14''.85.$$

$$\theta \sin \theta = 0.039473, \quad \log \frac{a + \cos \theta}{a + 1} = -0.2218488.$$

$$\log \frac{\cos \frac{1}{2}(\alpha - \theta)}{\cos \frac{1}{2}(\alpha + \theta)} = 0.6406695, \frac{\sin \theta}{\sin \alpha} \log \frac{\cos \frac{1}{2}(\alpha - \theta)}{\cos \frac{1}{2}(\alpha + \theta)} = -0.408267.$$

$$\frac{a}{m_1}(0.408267 - 0.2218488) = 0.407994.$$

$$y = \frac{2 \times 9000 \times 6060}{25000000} (0.039473 + 0.980198 - 1 + 0.407994)$$
  
= 1.86169 inches.

EXAMPLE 3. — Truncated parabolic bowstring, equation (317).  $h_1 = 300$ ,  $h_0 = 180$ ,  $x = \frac{1}{2}l = 1,200$ ,  $m^2 = \frac{h_1}{h_1 - h_0} = 2.5$ .

$$m(\frac{1}{2}l) = 1897.366, \ m(\frac{1}{2}l) + x = 3097.366, \ m(\frac{1}{2}l) - x = 697.366.$$

$$\log \frac{1}{2}ml = 3.2781512, \qquad \log (\frac{1}{2}ml + x) = 3.4909925,$$

$$\log (\frac{1}{2}ml - x) = 2.8434608.$$

Therefore (317) becomes

$$y = \frac{2.302585 \times 9000 \times 1200}{1.58114 \times 120 \times 25000000} (10812.88 + 1982.93 - 12439.70)$$
  
= 1.86695 inches.

EXAMPLE 4. — Chords of uniform slope.  $h_1 = 300$ ,  $h_0 = 180 = h$ ,  $\frac{1}{2}l = 1,200$ . Use equation (311).

$$y = \frac{2 \times 9000 \times 1200^{2}}{25000000 \times 120^{2}} \left\{ 300 - 180 \left( 2.302585 \log \frac{300}{180} + 1 \right) \right\}$$
  
= 2.0197 inches.

Example 5. — Circular bowstring.  $\frac{1}{2}l = 1,200, h_1 = 300, h_0 = 0$ . Use equation (328).

$$R = \frac{(\frac{1}{2}l)^2 + h_1^2}{2h_1} = 212.5 \text{ feet} = 2550 \text{ inches.}$$

$$\sin \theta = \frac{\frac{1}{2}l}{R} = 0.470588, \quad \theta = 28^{\circ} 4' 21''.$$

$$\cos \theta = \frac{R - h_1}{R} = 0.882353. \quad \text{In arc, } \theta = 0.501346.$$

$$\cos \alpha = -\cos \theta = \alpha = -0.882353.$$

$$\alpha = 180^{\circ} - 28^{\circ} 4' 21'' = 151^{\circ} 55' 39''.$$

$$\frac{1}{2}(\alpha - \theta) = 61^{\circ} 55' 39'', \quad \log \frac{\alpha + 1}{2[\cos \frac{1}{2}(\alpha - \theta)]^2} = -0.5757318.$$

$$y = \frac{2 \times 9000 \times 2550}{25000000} (0.23593 + 0.882353 - 1 + 2.302585 \times 0.882353 \times 0.5757318) = 2.36477 \text{ inches.}$$

Example 6. — Parabolic bowstring.  $\frac{1}{2}l = 1,200, h_1 = 300, h_2 = 0.$ 

By equation (319),

 $\theta \sin \theta = 0.23593.$ 

$$D = \frac{1.386295 \times 9000 \times 1200^2}{25000000 \times 300} = 2.39552 \text{ inches.}$$

EXAMPLE 7. — Girder sloping uniformly from centre to ends.  $\frac{1}{2}l = 1,200$ ,  $h_1 = 300$ ,  $h_0 = 0$ . By equation (313),

$$D = \frac{2 \times 9000 \times 1200^2}{25000000 \times 300} = 3.456 \text{ inches.}$$

Example 8. — Parabolic crescent.  $h_1 = 300$ ,  $h_0 = 0$ ,  $\frac{1}{2}l = 1,200$ .

The deflection in this case must be the same as that in the sixth example, for the parabolic bowstring.

.. 
$$D = 2.39552$$
 inches.

Example 9. — Girder like Fig. 53, sloping uniformly from centre to ends.  $h_1 = 300$ ,  $h_0 = 0$ ,  $\frac{1}{2}l = 1,200$ .

Deflection the same as in example 7, viz.,

$$D = 3.456$$
 inches.

Example 10. — Girder like Fig. 66, polygonal.

Find the deflection for each part having a uniform slope, separately, and add the results for the total central deflection, after correcting.

Take  $h_1 = 300$  at  $Z_4$ , and  $h_0 = 240$  at  $Z_6$ , the quarter-section. Then  $\frac{1}{2}l = 600$ , and equation (311) gives the deflection at  $Z_6$ , thus,

$$y = \frac{2 \times 9000 \times 600^{2}}{25000000 \times 60^{2}} [300 - 240(2.302585 \log \frac{800}{240} + 1)]$$
  
= 0.46408 inch.

Similarly, for the end quarter, equation (313) gives

$$D = \frac{2 \times 9000 \times 600^2}{25000000 \times 240} = 1.08 \text{ inches.}$$

But, before adding these results, we must find, as in article 67, how much the free end of the semi-beam is deflected by reason of the bending of the part between  $Z_4$  and  $Z_6$ ; that is, we must add to 0.46408 the quantity  $\frac{1}{4}l \times \tan \alpha = 600 \tan \alpha$ 

$$=600\frac{dy}{dx}.$$

From (311),

$$\frac{dy}{dx} = \tan \alpha = \frac{2B_1 l}{E(h_0 - h_1)} \log_e \frac{h}{h_1} = 0.0016066,$$

$$600 \times 0.0016066 = 0.96398 \text{ inch,}$$

.. Total deflection = 0.46408 + 0.96398 + 1.08 = 2.50806 inches,

which is greater than the deflection found in example 6, for the parabolic bowstring; and it will be found that although the girder, Fig. 66, is deeper at the quarter-points than the parabolic bow of example 6, yet at the \{\frac{1}{8}\} and \{\frac{3}{8}\} points the latter is the deeper.

In like manner may we proceed in all cases of irregular forms, whether there be two or more changes of slope; but, in general, we may use the formulæ already found for regular forms, with sufficient accuracy, always choosing the one most fitting for the case in hand.

94. We may arrange the results found in these examples according to the amount of the deflection, and thus the more clearly perceive the effect of form upon the bending of girders of uniform strength. All the girders here represented are 200° feet in length if supported at both ends, or 100 feet long if semi-girders; the deflection being the same in either case.

Since in all the formulæ the deflection varies directly as  $\frac{B_I}{E} = \frac{\frac{1}{2}(C_I + T_I)}{E}$  we may find the deflection of girders of the same dimensions, but of other material than wrought-iron, by substituting for E the proper value taken from Table II., and for  $C_I$  and  $T_I$  the allowed unit strain.

If, for pine,  $T_1 = 1,200$ ,  $C_1 = 552$ , E = 1,460,000, then  $B_1 = 876$ , and  $\frac{B_1}{E} = 0.0006$ , but for wrought-iron  $\frac{B_1}{E} = 0.00036$ ; hence, for a girder of uniform strength and of given span and

height, the deflection, if the material is pine, will be five-thirds of the deflection were the material wrought-iron; that is, allowing  $C_1$  and  $T_2$  the above values.

If the compressed chord be of pine,  $C_1 = 552$ , and the other of wrought-iron,  $T_1 = 10,000$ , and if  $E = 13,230,000 = \frac{1}{2}(25,000,000 + 1,460,000)$ , we have  $B_1 = 5,276$ ,  $\frac{B_1}{E} = 0.0004$ . Hence a combination of pine and wrought-iron gives a deflection  $\frac{4}{3.6} = \frac{10}{9}$  times that due wrought-iron alone, with these unit strains.

Were the compressed chord of cast-iron, for which  $C_1 = 15,000$ , while the other chord is of wrought-iron,  $T_1 = 10,000$ , and  $E = \frac{1}{2}(25,000,000 + 12,000,000) = 18,500,000$ , we should have  $B_1 = 12,500$ ,  $\frac{B_1}{E} = 0.00067567$ , and the deflection would be 1.877 times that of the girder of same size in wrought-iron.

95. By inspecting the following table, we see that for open girders of the same central height, same length, and of uniform strength, the total deflection is NEARLY in the inverse ratio of the areas of the figures of the girders.

This is exactly the ratio of the deflections in case of the girder of uniform height and of that sloping uniformly to a point: viz., ratio of areas,  $\frac{2}{1}$ ; ratio of deflections,  $\frac{1}{2}$ . We may, therefore, without appreciable error, employ this principle in finding the total deflection of open girders of uniform strength and variable height.

Examples. — Deflection of Open Webbed Girders of Uniform Strength. Length =  $l=2\infty$  feet, central height =  $k_1=25$  feet.

	Material.	Wrought- Iron.	Pine.	WrtIron and Pine.	Wrt. and Cast Iron.	_1	
	B <sub>1</sub> E	9,000	876 1,460,000	5,276 13,230,000	12,500	Equa- tion.	
	Form.	Description.	Def., ins.	Def., ins.	Def., ins.	Def., ins.	
1.	25	Uniform } height }	1.728	2.88	1.92	3.24	(308)
:2.	25 25 15 25	One chord circular	1.86169	3.1028	2.0685	3-494	(327)
:3.	25 15 25 15	One chord } parabolic }	1.86695	3.1116	2.0744	3.504	(317)
4.	25 15 25 15 25 15	Uniform } slope }	2.0197	3.366	2.2441	3.791	(311)
<b>15</b> .	25	Circular Bowstring	2.36477	3.9413	2.6275	4-438	(328)
6.	25	Parabolic } curves }	2.39552	3.9925	2.6617	4.496	(319)
7.	25 26 26	Uniform } slope }	3.456	5.76	3.84	6.48	(313)
8.	25 20	Polygonal } chord }	2.50806	4.1801	2.7867	4-703	(311)

We give below, the deflections of girders of wrought-iron for the eight cases just tabulated, but now computed by this Method of Areas:—

No.	Area of One-Half Girder.  2500 square feet.			Deflection.  1.7280 inches.		Deflection by Formulæ.  1.72800 inches.	
I							
2	2167	- 66	"	1.9931	66	1.86169	44
3	2167	66	66	1.9938	66	1.86695	66
4	2000	66	"	2.1600	66	2.01970	66
-	1737	66	"	2.4871	66	2.36477	"
5 6	1667	66	66	2.5920	46	2.39552	"
7	1250	44	44	3.4560	"	3.45600	66
8	1625	"	44	2.6585	66	2.50806	46

The deflections of such girders as those shown in Figs. 19, 20, 29, 30, 31, 32, 33, 34, 39, 40, 53, 54, 55, etc., are therefore easily found by the method of areas.

It should be noticed that in the preceding table of deflections of the same girder in different materials, a factor of safety equal to 10 has been allowed for pine, while 5 is the factor allowed for wrought and for cast iron.

The modulus of elasticity for cast-iron, E = 12,000,000, is so small, that, in spite of its large resistance to compression,  $C_1 = 15,000$ , the open beam made of wrought and cast iron, and of uniform strength, has greater deflection than that of wrought-iron alone, and, indeed, greater than that of pine alone, with the low unit strain here allowed.

**96.** Finally, if the beam be of uniform strength, but have a continuous web, the formulæ already deduced for girders of uniform strength and of open web may be employed by assigning to  $B_1$  its proper value derived from Table II.

Example 1. — Plate girder of uniform strength and uniform height, wrought-iron. Take the length l=50 feet = 600 inches, height h=5 feet = 60 inches; the girder being supported at both ends.

By Table II., B=42,000= breaking unit strain for plate beams. Allowing a safety factor of 5, we have  $B_1=8,400$ ; and calling E=25,000,000, and putting  $\frac{1}{2}l$  for l in equation (308), there results the central deflection,

$$D = \frac{8400 \times 300^2}{25000000 \times 60} = 0.504 \text{ inch.}$$

EXAMPLE 2. — Take a plate girder of the same length, 50 feet, and same central height, 5 feet, but sloping uniformly from centre to ends, where the height is 2 feet.

Then, if the girder is of uniform strength, we have, from equation (311),

$$y = \frac{2 \times 8400 \times 300^{2}}{25000000 \times (60 - 24)^{2}} [60 - 24(2.302585 \log \frac{60}{24} + 1)]$$
  
= 0.65376 inch.

EXAMPLE 3. — Cast-iron beam of uniform strength, and height h = 3 feet = 36 inches,  $\frac{1}{2}l = 6$  feet = 72 inches. Take  $B_1 = \frac{88350}{6} = 7,650$ , E = 17,000,000 (Table II.).

Then, by (308),

$$D = \frac{7650 \times 72^2}{17000000 \times 36} = 0.0648 \text{ inch.}$$

Example 4. — If this cast-iron beam of uniform strength slope uniformly from centre to ends, where h = 12 inches, then, by (311),

$$y = \frac{2 \times 7650 \times 72^2}{17000000 (12 - 36)^2} [36 - 12(2.302585 \log \frac{86}{12} + 1)]$$
  
= 0.0876 inch.

EXAMPLE 5. — Oak beam of uniform strength and height. Take  $\frac{1}{2}l = 120$  inches, k = 18 inches,  $B_1 = \frac{10600}{10}$ , E = 2, 150,000; then (308) gives

$$D = \frac{1060 \times 120^2}{2150000 \times 18} = 0.3944 \text{ inch.}$$

Example 6. — If this oak beam of uniform strength slope uniformly from centre to ends, where h = 12 inches, then, by (311),

$$y = \frac{2 \times 1060 \cdot \times 120^{2}}{2150000 \times (12 - 18)^{2}} [18 - 12(2.302585 \log \frac{18}{12} + 1)]$$
  
= 0.4474 inch.

EXAMPLE 7. — Beam of Bessemer hammered steel, uniform strength.  $\frac{1}{2}l = 72$  inches; height at centre,  $h_1 = 20$  inches, at ends,  $h_0 = 10$  inches. Take  $B_1 = \frac{128088}{5} = 25,616$ , E = 31,000,000 (Table II.).

Then deflection at centre is, from (311),

$$y = \frac{2 \times 25616 \times 72^{2}}{31000000(10 - 20)^{2}} [20 - 10(2.302585 \log \frac{20}{10} + 1)]$$
= 1.1196 inches.

For same beam of wrought-iron,

$$y = \frac{2 \times 9000 \times 72^{2}}{25000000(10 - 20)^{2}} [20 - 10(2.302585 \log \frac{20}{10} + 1)]$$
  
= 0.4878 inch,

which is less than half the deflection of the same beam in steel.

But if we suppose this beam to be of rectangular cross-section, and to bear a concentrated weight, W, at its centre, where the height is  $h_1 = 20$  inches, and the thickness b = 2

inches, the length being l = 144 inches, then, from equations (46) and (160), we have moment at centre,

$$M = \frac{1}{4}Wl = \frac{1}{8}Bbh^2 = \frac{1}{8}B_1bh^2$$
 for safety,

$$\therefore W = \frac{2}{3} \times \frac{2 \times 20^2}{144} B_1,$$

 $W = 3.7037 \times 25616 = 94874$  pounds for steel,

 $W = 3.7037 \times 9000 = 33333$  pounds for wrought-iron.

Hence, under the assumed unit strains, the steel beam bears  $\frac{25616}{9000} = 2.8462$  times the weight at the centre of the wroughtiron beam of the same dimensions, while the deflection of the steel beam is  $\frac{1.1196}{0.4878} = 2.2953$  times that of the wrought-iron beam; that is, what is shown in all the formulæ, the weight W varies directly with the unit strain  $B_1$ , while for the same unit strain the deflection varies inversely as the modulus of elasticity, E.

Therefore in the present case, so far as deflection is concerned, the advantage of steel over wrought-iron, under same load, is  $\frac{2.8462}{2.2953} = \frac{31}{25}$ , which is the simple ratio of the moduli of elasticity.

97. The thickness, b, of a continuous webbed girder of uniform strength at any rectangular section of given height, h, may be found, in general, by equating the moment, M, due the external forces, to the moment of resistance, R, of the internal forces of the beam at the given section, and solving with respect to b.

For a beam of rectangular cross-section, bearing a concentrated load at its centre, equations (45) and (160) give  $M = \frac{1}{2}Wx = \frac{1}{6}B_1bh^2$ .  $B_1 =$  allowed unit strain.

$$\therefore \quad b = \frac{3Wx}{B_1h^2},\tag{329}$$

where, if the height, h, be uniform, b varies as x; making the horizontal projection or ground plan of each half of the beam a triangle with a vertex at the end of the beam, where b = x = 0, and a base at the beam's centre, where  $b = \frac{3}{2} \frac{Wl}{b_1 h^2}$ .

Example. — Oak beam of uniform strength, and height h = 15 inches, length = 15 feet, weight applied at centre = W = 4,000 pounds allowed unit strain  $= \frac{10600}{10} = 1,060$  pounds per square inch. What must be the thickness of this beam at the centre?

Here  $x = \frac{1}{2}l = 90$  inches,

$$b = \frac{3 \times 4000 \times 90}{1060 \times 15^2} = 4.53 \text{ inches.}$$

It must be remembered that wherever the moment becomes zero, causing b, the thickness of the beam, to vanish by the formulæ, we must, nevertheless, have at all such points sufficient material to resist, with the proper margin of safety, the shearing-strain which may there be developed, and the re-action of the supports.

In this example the shearing-strain at each end of this beam is  $\frac{4000}{2} = 2,000$  pounds. Now, by Table I., the ultimate resistance to shearing is, for oak, across the grain, 4,000 pounds;

one-tenth of which is 400 pounds, to be allowed to each square inch of the vertical section at each end.

Therefore  $\frac{2000}{400} = 5$  square inches of section at least the beam must have at each end; that is, the depth being 15 inches, the thickness is  $\frac{1}{8}$  inch. But there is another consideration to be attended to; viz., the bearing-surface at the ends must be sufficient to resist with safety and permanence the pressure coming upon it.

This beam as now estimated is  $\frac{1}{8}$  inch thick at each end, and 4.53 inches at its centre. Hence it must have 8.903 inches of its length at each end upon the support, in order to secure a bearing of  $3\frac{1}{8}$  square inches, required for 2,000 pounds with an allowed unit strain of 600 pounds to the square inch, in compression.

Again, a beam so thin at the ends would lack lateral stiffness unless it were walled in.

In practice, therefore, even when it is desired to use the least material possible, it is customary to make those parts of a beam which theoretically, or rather, by formula, are almost nothing, of such size as a just regard to all these requirements, as well as to the good appearance of the structure, may demand.

Let it not be inferred that theory and practice are at variance here, for such is not the case. The equations which determine the thickness of the beam do not pretend to take into the account all the conditions affecting the sufficiency of the beam for its purpose. And hence the theory is not complete till the modifying conditions are introduced.

98. If the beam of uniform strength be loaded uniformly with w units of weight to the unit of length, we have, from equations (49) and (160),

$$\frac{1}{8}w(l-x)x=\frac{1}{8}B_1bh^2,$$

putting  $B_1$  for  $B_2$ , and the cross-section being rectangular;

$$\therefore \quad b = \frac{3w(l-x)x}{B_1h^2},\tag{330}$$

which is the thickness of the beam at any point, x, measured from the end. When h is constant, (330) is the equation of a parabola; the vertex being at the end of the beam.

Thickness at end 
$$= b = 0$$
.  $x = 0$ .

Thickness at centre = 
$$b = \frac{3wl^2}{4B_1h^2}$$
.  $x = \frac{1}{2}l$ .

Horizontal projection, two parabolas.

Example. — Oak beam, uniform strength. Height uniform = h = 15 inches, length l = 180 inches,  $B_1 = \frac{10600}{10} = 1,060$ ,  $w = \frac{8000}{180} = 44\frac{4}{5}$  pounds per inch.

Then thickness at centre is

$$b = \frac{3 \times 400 \times 180^2}{4 \times 9 \times 1060 \times 15^2} = 4.53$$
 inches.

99. If the cross-section of the beam of uniform strength be of either form, Fig. 91, then, by assigning values to three of the dimensions, h,  $h_1$ , b,  $b_1$ , we may, from equation (161) and the equation expressing the moment due the given load, find the fourth dimension of the cross-section, which, therefore, becomes known at every point.

In like manner may we determine any one dimension of any cross-section whose moment of resistance, R, is known.

EXAMPLE. — Take a tubular plate girder of the dimensions given in example 1, article 96; viz., l = 50 feet, h = 5 feet,  $B_1 = 8,400$ , uniform strength and height. Cross-section as in Fig. 91, where let b = 12 inches,  $b_1 = 11\frac{1}{4}$  inches; the side plates being  $\frac{8}{3}$  inch thick each, h = 60 inches.

From (49) and (161), we have

$$\frac{1}{2}w(l-x)x = \frac{1}{6}B_1 \frac{bh^3 - b_1h_1^3}{h},$$

$$\therefore h_1 = \left(\frac{bh^3}{b_1} - \frac{3hw(l-x)x}{B_1b_1}\right)^{\frac{1}{3}},$$
(331)

equal to 58 inches if wl = 123,508 pounds, the total uniform load on beam, and  $x = \frac{1}{2}l = 300$  inches.

At the centre, therefore, the top and bottom plates must have the thickness of I inch each; while at the ends, where x = 0, (331) gives

$$h_1 = h \left(\frac{12}{11.25}\right)^{\frac{1}{3}} = 1.02174h = 61.3044$$
 inches,

which renders  $h - h_1 = -1.3044$  inches negative, showing that the cross-section of the side plates is more than sufficient at the ends to resist the moment.

We may find at what distance from either end of this beam the top and bottom plates begin to be needed, by putting  $h_r = h = 12$  in (331), and finding x. This gives x = 69.19 inches, for which the side plates alone are sufficient if properly braced laterally. Now, the shearing-strain at each end of the beam supporting this load is  $\frac{1}{2} \times 123,508 = 61,754$  pounds; and, calling the allowed shearing-strain 8,000 pounds to the square inch, we require  $\frac{61754}{8000} = 7.72$  inches in cross-section of the two plates, whereas we have  $2 \times \frac{3}{8} \times 60 = 45$  square inches. But

in order to have sufficient bearing-surface on the abutments, allowing the iron to bear 8,000 pounds to the square inch in compression also, the beam must be supported for at least 7.72

$$\frac{7.72}{2 \times \frac{3}{2}}$$
 = 10.3 inches of its length at each end.

The semi-girder of uniform strength and continuous web is to be treated in the same manner as the girder just considered when we seek its variable cross-section.

100. Beam of Uniform Strength fixed Horizontally at Both Ends. — By definition the beam of uniform strength is equally efficient at all sections to resist the strains generated by the external forces. Hence, when this beam is horizontally fixed at both ends, and loaded with a concentrated or with a continuous load, the points of contrary flexure are, for any style of beam or girder, practically midway between the centre of gravity of the load and the ends of the girder; since there is as much reason for their being on one side of this midway point as there is for their being upon the other side of it, and And the beam of uniform strength is such only with reference to a particular mode of loading. That is, if the unit strain is uniform throughout the girder for a given position of the load, a change in the position of the load causes a change in the relative values of the total strains in the members or parts of the girder, and therefore a change in the unit strain on each member, if, as is assumed, the cross-sections of the members be not changed.

In general, we have for any girder, from equations (184) and (187),

Moment due internal forces, 
$$M_x = \frac{2B_1I}{h} = \frac{2B_1Sr^2}{h}$$
, (332)

where  $B_1$  = allowed unit strain in bending, S = area of any cross-section, r = radius of gyration of the section about its neutral axis, h = height of section.

By equating the last member of (332) to the known moment due the external forces applied to the girder, any *one* of the four quantities  $B_r$ , S, r, k, may be found. But, when the girder is fixed at one or both ends, we need to know the point or points of contrary flexure, in order to determine the end moments.

tot. For the girder of uniform height and strength, fixed at both ends, it follows from the uniformity of the unit strain and height, which causes a uniformity of curvature, that, as already stated, each point of contrary flexure is sensibly midway between the centre of gravity of the load and the corresponding end of the girder.

Assuming that the height and strength are uniform, and that, for any required form of cross-section, the necessary variation in its area is attained by varying the thickness of the beam only, we shall have, in (332), r, h, and B, constant, so that the variable area, S, may be found at once for any section of the beam; and from S the thickness is to be determined.

Both Ends, and bearing a Concentrated Weight, W, at the Distance a' from the Left End.—The moment at any point between the weight and left end of the beam, that is, when x is not greater than a', is given by equations (40), (93), and (332), thus,

$$M_x = W \frac{l - a'}{l} x - \frac{M_1 - M_2}{l} x + M_1 = \frac{2B_1 S r^2}{h}.$$
 (333)

Now,  $M_x = 0$  when  $x = \frac{1}{2}a'$ ,

$$\therefore o = W^{l - d'} a' - \frac{a'}{2l} (M_1 - M_2) + M_1.$$
 (334)

Also, when x is not less than a', we have, from (43), (93), and (332),

$$M_x = W \frac{l - x}{l} a' - \frac{M_1 - M_2}{l} x + M_1 = \frac{2B_1 S r^2}{h}.$$
 (335)

 $M_x = 0$  when  $x = \frac{1}{2}(l + \alpha)$ ,

$$\therefore \quad 0 = W^{\frac{l-a'}{2l}}a' - \frac{1}{2}(M_1 - M_2) - \frac{a'}{2l}(M_1 - M_2) + M_1. \quad (336)$$

From (334) and (336), we find

$$M_1 = M_2 = -\frac{1}{2}W\frac{a'}{l}(l-a').$$
 (337)

That is, the end moments are equal and negative for any given position of the load.

Eliminating  $M_1$  and  $M_2$  from (333) and (335), we obtain

$$x \leq a', \qquad M_x = W \frac{l - a'}{l} (x - \frac{1}{2}a') = \frac{2B_t S r^2}{h}, \qquad (338)$$

$$S = \frac{Wh(l - a')}{2B_1 l r^2} (x - \frac{1}{2}a'). \tag{339}$$

$$x \equiv a', \qquad M_x = W \frac{a'}{l} \left( \frac{l + a'}{2} - x \right) = \frac{2B_1 S r^2}{h}, \qquad (340)$$

$$S = \frac{Wha'}{2B_1 lr^2} \left( \frac{l + a'}{2} - x \right) \tag{341}$$

EXAMPLE 1. — If the varying cross-section is a rectangle of the breadth b, and constant height k, we have  $r^2 = \frac{1}{12}k^2$ , and (339) and (341) become

$$x \leq a',$$
  $S = bh = \frac{6W(l-a')}{B_1 lh}(x-\frac{1}{2}a'),$  (342)

$$b = \frac{6W(l-a')}{B_1 l h^2} (x - \frac{1}{2}a'). \tag{343}$$

$$x \equiv a',$$
  $S = bh = \frac{6Wa'}{B_1lh}\left(\frac{l+a'}{2}-x\right),$  (344)

$$b = \frac{6Wa'}{B_1 lh^2} \left( \frac{l + a'}{2} - x \right). \tag{345}$$

If, further, the weight, W, is at the centre of the girder,  $a' = \frac{1}{2}l$ , and when

$$x \leq a',$$
  $b = \frac{3W}{B_1h^2}(x - \frac{1}{4}l).$  (346)

$$x \overline{\nearrow} a', \qquad b = \frac{3W}{B_1 h^2} (\frac{3}{4}l' - x).$$
 (347)

In (346), for 
$$x = 0$$
,  $b = b_1 = -\frac{3W7}{4B_1h^2}$ , at left end.

$$x = \frac{1}{4}l$$
,  $b = 0$ , at quarter point.

$$x = \frac{1}{2}l$$
,  $b_c = \frac{3W7}{4B_1h^2}$ , at centre.

In (347), for 
$$x = \frac{1}{2}l$$
,  $b_c = \frac{3Wl}{4B_1h^2}$ , at centre.

$$x = \frac{3}{4}l$$
,  $b = 0$ , at quarter point.

$$x = l$$
,  $b = b_2 = -\frac{3Wl}{4B_1h^2}$ , at right end.

If the beam is of oak, and  $B_1 = \frac{1}{10}B = 1,060$  pounds, E = 2,150,000, l = 180 inches, k = 15 inches, W = 4,000 pounds, then  $b_1 = b_2 = -b_c = -\frac{3 \times 4000 \times 180}{4 \times 1060 \times 15^2} = -2.264$  inches; the algebraic sign only indicating the direction of the inclination of the vertical planes forming the sides, to the vertical longitudinal plane of the beam.

Fig. 99 shows this beam thus loaded, in plan and elevation.

It is evident that the deflection of the part  $BD = \frac{1}{2}l$ , or of the part  $AB = \frac{1}{4}l$ , as a semi-beam, is equal to the deflection of a beam of uniform strength and height supported but not fixed at the points B and D, and bearing the concentrated weight W. But, by equation (307), the deflection of the part AB or BD is, since for x we must put  $\frac{1}{4}l$ ,

$$D = \frac{B_1 l^2}{16Eh}.$$

$$C$$

$$D$$

$$E$$
Fig. 99.

Therefore the total deflection at C, the centre of AE, is

$$2D = \frac{Bl^2}{8Eh},\tag{348}$$

equal to  $\frac{1060 \times 180^2}{8 \times 2150000 \times 15} = 0.1331$  inch in the present case.

At the points of contrary flexure, where b = 0, the beam, of course, must be enlarged, to resist with safety the shearing-strains.

The shearing-strain at each of these points is now  $\frac{1}{2}W = 2,000$  pounds. By Table I., article 42, the ultimate shearing-strength of oak across the grain is 4,000 pounds to the inch; or the working-strength is 400 pounds to the square inch of cross-section.

We require, therefore, at least  $\frac{2000}{400} = 5$  square inches of area at each point of contrary flexure; that is, the beam, being 15 inches deep, must be at least  $\frac{1}{3}$  inch thick at these points, even when restrained from moving laterally.

103. Beam of Uniform Strength, Height, and Load, fixed Horizontally at Both Ends. Rectangular Cross-Section.

Equations (49) and (93) give

$$M_x = \frac{1}{2}w(l-x)x - \frac{M_1 - M_2}{l}x + M_1 = \frac{1}{6}B_1bh^2$$
. (349)

Make x = 0, then

$$M = M_2 = \frac{1}{6}B_1b_1h^2.$$

But, as in article 102, b = 0 when  $x = \frac{1}{4}l$  or  $\frac{3}{4}l$ ,

$$\therefore \ \frac{1}{2}w(l-\frac{1}{4}l)\frac{1}{4}l+M_{1}=0,$$

$$M_1 = M_2 = -\frac{8}{32}wl^2,$$

$$M_c = \frac{1}{2}w(l - \frac{1}{2}l)\frac{1}{2}l - \frac{3}{32}wl^2 = \frac{1}{32}wl^2,$$

$$M_x = \frac{1}{2}w(l-x)x - \frac{3}{82}wl^2 = \frac{1}{6}B_1bh^2.$$
 (350)

$$b = \frac{3w}{B_1 h^2} [(l-x)x - \frac{3}{16}l^2]. \tag{351}$$

When x = 0, or x = l, (351) gives

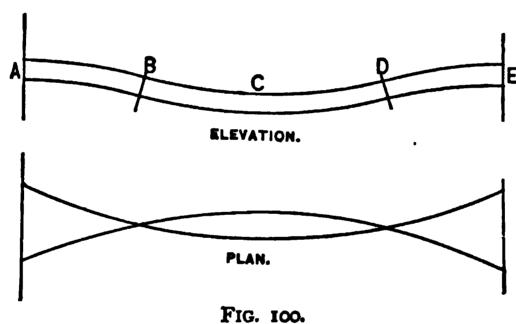
$$b = b_1 = b_2 = -\frac{9wl^2}{16B_1h^2},$$

which is the width of beam at either fixed end;

$$b_c = \frac{3w}{B_1h^2} \left\{ \left( l - \frac{1}{2}l \right) \frac{1}{2}l - \frac{3}{16}l^2 \right\} = \frac{3wl^2}{16B_1h^2},$$

which is the width of the beam at centre, being one-third of the width at either end.

Equation (351) being that of a parabola with respect to the variables b and x, the plan of this uniformly loaded beam, of uniform strength and height, fixed at its ends, is shown in Fig. 100.



Example. — Oak beam. l = 180 inches, h = 15 inches, ends fixed.

Take  $B_1 = 1,060$ , E = 2,150,000,  $w = \frac{8000}{180} = 44\frac{4}{9}$  pounds to the linear inch.

Then

$$M_1 = M_2 = -\frac{3}{3^2} \times \frac{400}{9} \times 180^2 = -135000$$
 inch-pounds.  
 $M_6 = 45000$  inch-pounds.

$$b_1 = b_2 = -\frac{9 \times \frac{400}{8} \times 180^2}{16 \times 1060 \times 15^2} = -3.3962$$
 in. = thickness at ends.

 $b_c = 1.1321$  inches, at centre.

The maximum deflection is evidently given, as in article 102, by equation (348), and is

$$D = 0.1331$$
 inch.

the Unfixed End of a Beam of Uniform Strength, fixed at the Right End, but simply supported at the Left. — The point of contrary flexure must be at the distance  $x_0 = \frac{1}{2}(l + \alpha')$  from the unfixed end, in order that the greatest positive moment may be equal to the greatest negative moment.

Equations (43) and (93) apply, giving, since  $M_1 = 0$ ,

$$M_x = W \frac{l - x}{l} a' + M_2 \frac{x}{l} = \frac{1}{6} B_1 b h^2 \qquad (352)$$

if the cross-section be rectangular, and  $x \ge d$ .

For 
$$x = l$$
,  $M_2 = \frac{1}{6}B_1b_2h^2$ .  
For  $x = \frac{1}{2}(l + a')$ ,  $M_2 = -W\frac{a'(l - a')}{l + a'}$  when  $b = 0$ .  
For  $x = a'$ ,  $M_{a'} = W\frac{a'(l - a')}{l + a'}$ .  

$$b = \frac{6W}{B_1h^2} \left\{ \frac{a'}{l + a'}(l + a' - 2x) \right\}. \tag{353}$$
When  $x = l$ ,  $b = b_2 = -\frac{6W}{B_1h^2} \left( \frac{a'(l - a')}{l + a'} \right)$ .

When 
$$x = a'$$
, 
$$b = \frac{6W}{B_1h^2} \left(\frac{a'(l-a')}{l+a'}\right).$$

Which shows that the width at D, Fig. 101, is the same as the width at C for h constant.

When x = a', use (40) and (93), giving

$$M_x = W^{\frac{l}{l} - \alpha'} x + M_2^{\frac{x}{l}} = \frac{1}{6} B_1 b h^2.$$
 (354)

To find the lowest point, E, in the curve, Fig. 101, we equate the deflection,  $D_1$ , between the lowest point and left end of the beam, to the total deflection,  $D_2 + D_3$ , between the same point and the right end of the beam, and solve the equation  $D_1 = D_2 + D_3$ .

For the length AE = z, (307) gives

$$D_{1} = \frac{B_{1}z^{2}}{Eh}.$$

$$EB = \frac{1}{4}(l + a') - s, \qquad D_{2} = \frac{B_{1}\left(\frac{l + a'}{2} - s\right)^{2}}{Eh}.$$

$$BD = \frac{1}{2}(l - a'), \qquad D_{3} = \frac{B_{1}\left[\frac{1}{2}(l - a')\right]^{2}}{Eh}.$$

$$\therefore s^{2} = \left(\frac{l + a'}{2} - s\right)^{2} + \frac{1}{4}(l - a')^{2},$$

$$s = \frac{l^{2} + a'^{2}}{2(l + a')}.$$

$$D_{1} = D_{2} + D_{3} = \frac{B_{1}(l^{2} + a'^{2})^{2}}{4Eh(l + a')^{2}}, \qquad (355)$$

which is the deflection at the lowest point, E.

EXAMPLE. — Given W = 4,000 pounds at the distance  $a' = \frac{1}{8}l$  from the unfixed end, A, Fig. 101; l = 180 inches; h = 15 inches = uniform height of beam;  $B_1 = \frac{1}{10}B = 1,060$  pounds = working inch strain for oak; E = 2,150,000. Crosssection rectangular.

Then width of beam is,

At left end, (354),

$$x = 0, \qquad b = 0.$$

At the weight, (353),

$$x = \frac{1}{8}l$$
,  $b = \frac{4000 \times 180}{1060 \times 15^2} = 3.019$  inches.

(353),

$$x=\frac{2}{3}l, \qquad b=0.$$

At fixed end, (353),

$$x = l$$
,  $b = -\frac{4000 \times 180}{1060 \times 15^2} = -3.019$  inches;

the negative sign simply showing that the lines cd,  $c_rd_r$ , have crossed somewhere between  $x = \frac{1}{8}l$  and x = l.

Moment at fixed end,

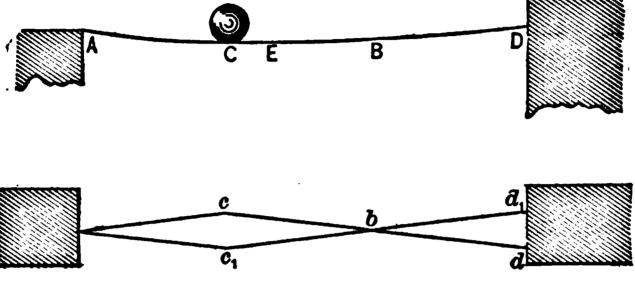
$$M_2 = -4000 \frac{\frac{1}{3} \times \frac{2}{3}}{1 + \frac{1}{3}} = -120000.$$

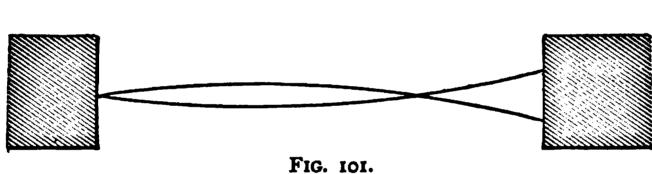
Moment at the weight,

 $M_{a'} = 120000$  inch-pounds.

The deflection at the lowest point, E, is given by (355),

$$D_1 = \frac{1060 \times (\frac{10}{8})^2 \times 180^2}{4 \times 2150000 \times 15 \times (\frac{4}{8})^2} = 0.1849 \text{ inch.}$$





105. Continuous Uniform Load, wl, on a Beam of Uniform Strength, fixed at the Right End, and simply supported at the Left, Fig. 101.—The figure shows the curvature and the plan, when the section is rectangular; and equations (49) and (93) give, since  $M_1 = 0$ ,

$$M_x = \frac{1}{2}w(l-x)x + M_2\frac{x}{l} = \frac{1}{8}B_1bh^2$$
 (356)

for these conditions.

For 
$$x = l$$
,  $M_2 = \frac{1}{6}B_1b_2h^2$ ,  $b_2 = -\frac{wl^2}{B_1h^2}$ .  
For  $x = \frac{2}{4}l$ ,  $M_2 = -\frac{1}{6}wl^2$ ,  $b = 0$ .

For 
$$x = \frac{1}{8}l$$
,  $M = \frac{1}{18}wl^2$ ,  $b = \frac{wl^2}{3B_xh^2}$ .
$$b = \frac{w}{B_xh^2}(2l - 3x)x. \tag{357}$$

Hence the breadth at C is one-third of that at D when the height is uniform, as seen in the parabolic plan, Fig. 101, derived from equation (357).

The deflection at E is given, by (355), for this case also, provided we put  $\frac{1}{3}l$  for  $\alpha'$ .

Example. — Let l = 180 inches, h = 15,  $B_1 = \frac{1}{10}B = 1,060$  pounds = working unit strain for bending oak, E = 2,150,000. Cross-section rectangular; load wl = 8,000 pounds uniformly distributed continuously, 44% pounds to the inch.

From (356) and (357),

$$x = 0$$
,  $M_1 = 0$ .  
 $x = \frac{1}{8}l$ ,  $M = \frac{1}{18} \times \frac{400}{8} \times 180^2 = 80000$  inch-pounds.  
 $x = \frac{2}{8}l$ ,  $M = 0$ .  
 $x = l$ ,  $M_2 = -\frac{1}{8} \times \frac{400}{8} \times 180^2 = -240000$  inch-pounds.

Width at left end,

$$b = 0$$
,

by (357).

Width at 1/4,

$$b = \frac{\frac{400}{9} \times 180^2}{3 \times 1060 \times 15^2} = 2.0126$$
 inches.

Width at  $\frac{3}{8}$ ,

$$b = 0.$$

Width at right end,

 $b_2 = 6.0378$  inches.

Put  $\frac{1}{8}l$  for  $\alpha'$  in (355), and find the deflection D=0.1849 at lowest point.

106. Beam of Uniform Strength and Uniformly Varying Height, fixed at Both Ends. — The end moments  $M_1 = M_2$  are determined for this case as in articles 102 and 103, for the same kind of load.

To find the deflection of this beam, we may regard it as composed of four semi-beams, Fig. 102.

1st, AB, fixed at A; deflection  $D_1$ .

2d, BC, fixed at C, the lowest point; deflection  $D_2$ .

3d, CE, fixed at C, the lowest point; deflection  $D_3$ .

4th, EF, fixed at F; deflection  $D_4$ .

Now we must have  $D_1 + D_2 = D_3 + D_4$ , from which the lowest point and its deflection may be found.

Example 1. — Take one-half of the girder shown in Fig. 65, and suppose the ends of this half to be immovably fixed. Call the length l = 100 feet = 1,200 inches; and height at left end,  $h_0 = 180$  inches; height at right end,  $h_1 = 300$  inches. Let the girder be of wrought-iron, and, as in article 93, take  $B_1 = \frac{1}{2}(C_1 + T_1) = 9,000$  pounds, E = 25,000,000; and suppose the load to be a concentrated weight, W = 200,000 pounds at the centre, no account being here taken of the girder's own weight.

By article 100, the points of contrary flexure are  $\frac{140}{2} = 25$  feet from the centre of the beam; and from equation (337), since  $a' = \frac{1}{2}l$ ,

$$M_1 = M_2 = -M_c = -\frac{1}{2} \times 200000 \times \frac{1}{2} \times \frac{1}{2} \times 1200,$$
  
= -30000000 inch-pounds.

The area S of any cross-section on the left of the weight is given by (339), and on the right by (341). But these equations suppose the section of the top chord to be equal to that of the bottom chord in the same vertical plane of section, and at the centre give

$$S_c = \frac{200000 \times 240 \times 600 \times \frac{1}{4} \times 1200}{2 \times 9000 \times 1200 \times \frac{1}{4} \times 240^2} = 27.777 \text{ inches};$$

at left end, (339),

$$S_1 = \frac{200000 \times 180 \times 600 \times -\frac{1}{4} \times 1200}{2 \times 9000 \times 1200 \times \frac{1}{4} \times 180^2} = -37.037 \text{ inches};$$

at right end, (341),

$$S_2 = \frac{200000 \times 300 \times 600 \times \frac{1}{4} \times 1200}{2 \times 9000 \times 1200 \times \frac{1}{4} \times 300^2} = 22.222$$
 inches;

the negative sign indicating only a difference in the direction of the lateral faces of the chords, that is, change of slope laterally.

But if S' = area of section of chord in compression, S'' = area of section of chord in tension, we have

$$C_{\rm i}S' = P = \frac{H}{\cos \alpha}, \tag{358}$$

$$T_{\mathbf{I}}S'' = U = \frac{H}{\cos \beta},\tag{359}$$

$$H = M + h,$$

according to the notation and equations of article 49.

From which,

$$S' = \frac{M}{C_{h} \cos \alpha'} \tag{360}$$

$$S'' = \frac{M}{T_1 h \cos \beta}.$$
 (361)

Calling  $C_1 = 8,000$  pounds,  $T_1 = 10,000$  pounds,  $\alpha$  being the inclination of the top chord for all parts between the points of contrary flexure, while  $\beta = 0$ , and  $\beta$  being the slope of top chord for the remainder of the beam, while  $\alpha = 0$ , we have, at either end,

$$\tan \beta = \frac{25 - 15}{100} = 0.1, \quad \cos \beta = 0.99503;$$
  
 $\tan \alpha = 0, \quad \cos \alpha = 1.$ 

At centre,

$$\tan \alpha = 0.1$$
,  $\cos \alpha = 0.99503$ ;  
 $\tan \beta = 0$ ,  $\cos \beta = 1$ .

At left end, (361), area of top section,

$$S'' = \frac{30000000}{10000 \times 180 \times 0.99503} = 16.750 \text{ inches.}$$

At left end, (360), area of bottom section,

$$S' = \frac{30000000}{8000 \times 180 \times 1} = 20.832$$
 inches.

At centre, (360), area of top section,

$$S' = \frac{30000000}{8000 \times 240 \times 0.99503} = 15.703$$
 inches.

At centre, (361), area of bottom section,

$$S'' = \frac{30000000}{10000 \times 240 \times 1} = 12.500$$
 inches.

At right end, (361), area of top section,

$$S'' = \frac{30000000}{10000 \times 300 \times 0.99503} = 10.050$$
 inches.

At right end, (360), area of bottom section,

$$S' = \frac{30000000}{8000 \times 300 \times 1} = 12.500$$
 inches.

The totals are:—
At left end,

 $S_1 = 37.582$  inches;

at centre,

 $S_c = 28.203$  inches;

at right end,

 $S_2 = 22.550$  inches;

differing somewhat, as was to be expected, from the areas computed on the supposition of equal top and bottom sections.

The deflection for each part of this girder is given by (311). See Fig. 102.

1st, 
$$AB$$
, fixed at  $A$ ;  $h_1 = 180$  inches,  $h = h_0 = 210$ ,  $\frac{h_1}{h} = \frac{6}{7}$ ,  $h_0 - h_1 = 30$ , and we have

$$y = D_1 = \frac{2 \times 9000 \times 300^2}{25000000 \times 30^2} \left\{ 180 - 210 \left( 1 - 2.302585 \log \frac{7}{6} \right) \right\}$$
$$= \frac{9}{125} \times 2.3715 = 0.171 \text{ inch.}$$

4th, EF, fixed at F; 
$$h_1 = 300$$
,  $h = h_0 = 270$ ,  $h_0 - h_1 = -30$ ,  $\frac{h_1}{h} = \frac{10}{9}$ ,

$$y = D_4 = \frac{2 \times 9000 \times 300^2}{25000000 \times (-30)^2} \left\{ 300 - 270 \left( 2.302585 \log \frac{10}{9} + 1 \right) \right\}$$
$$= \frac{9}{125} \times 1.5528 = 0.112 \text{ inch.}$$

Now, in equation (311),  $\frac{l}{h_0 - h_x}$  is constant, since the length, l, varies as the height, l. Therefore, —

2d, 
$$BC$$
, fixed at  $C$ ;  $h = 210$ ,

$$y = D_2 = \frac{9}{125}(h_1 - 210 \times 2.302585 \log h_1 + 210 \times 2.302585 \log 210 - 210).$$
  
3d, *CE*, fixed at *C*;  $h = 270$ ,

• 
$$y = D_3 = \frac{9}{125}(h_1 - 270 \times 2.302585 \log h_1 + 270 \times 2.302585 \log 270 - 270),$$
 and

$$D_1 + D_2 = D_3 + D_4.$$

After making the substitutions, and reducing, we find

$$h_1 = 236.15$$
 inches,

which is the depth of the girder at lowest point, C. Hence distance of lowest point from left end is

$$10(236.15 - 180) = 561.5$$
 inches.  
 $D_2 = 0.108$  inch,  $D_3 = 0.167$  inch.  
 $D_1 + D_2 = D_3 + D_4 = 0.279$  inch at  $C$ .

Example 2. — Take the same girder as in the precedir example, but let the load, W = 200,000 pounds, be 75 feet fro the left end. Then the points of contrary flexure are, when  $x = \frac{2}{3}l$ , and  $x = \frac{2}{3}l$ , and by (337), since now  $a' = \frac{2}{3}l$ ,

$$M_1 = M_2 = -\frac{1}{4} \times 200000 \times \frac{3}{4} \times \frac{1}{4} \times 1200$$
  
= -22500000 inch-pounds.

At the weight,  $x = a' = \frac{4}{4}l$ , (338) gives

$$M = 200000 \times \frac{1}{4} \times \frac{8}{8} \times 1200 = 22500000$$
 inch-pounds.

At the centre,  $x = \frac{1}{2}$ ,

$$M_c = 200000 \times \frac{1}{2} \times \frac{1}{8} \times 1200 = 7500000$$
 inch-pounds.

At left end, by (361), area of top section,

$$S'' = \frac{22500000}{10000 \times 180 \times 0.99503} = 12.56$$
 inches.

At left end, by (360), area of bottom section,

$$S' = \frac{22500000}{8000 \times 180 \times 1} = 15.63$$
 inches.

At the weight, (360), area of top section,

$$S' = \frac{22500000}{8000 \times 270 \times 0.99503} = 10.47$$
 inches.

At the weight, (361), area of bottom section,

$$S'' = \frac{22500000}{10000 \times 270 \times 1} = 8.33$$
 inches.

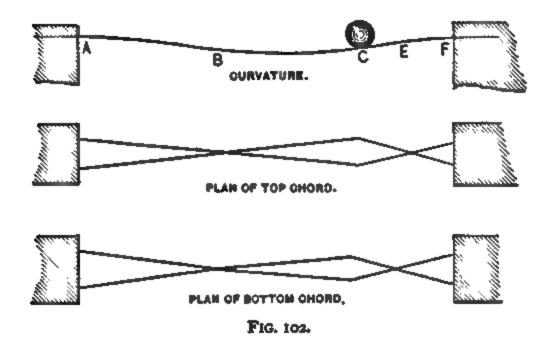
At right end, (361), area of top section,

$$S'' = \frac{22500000}{10000 \times 300 \times 0.99503} = 7.54$$
 inches.

At right end, (360), area of bottom section,

$$S' = \frac{22500000}{8000 \times 300 \times 1} = 9.38$$
 inches.

Since equations (338) and (340) are of the first degree with respect to M and x, and since k varies uniformly with x, we



have M and S in (360) and (361), varying uniformly from the ends of the girder and from the weight to the points of contrary flexure, as shown in plan of chords, Fig. 102, where the

depth of each chord is supposed to be uniform, and the varion in size of chord attained by varying the thickness only.

The deflection for each of the four semi-beams into wh the girder now becomes divided, is to be found as in the precing example, where the load was at the centre.

We now have, for the part  $AB = \frac{3}{8}l$ , fixed at A,  $h_1 = 1$ .  $h = h_0 = 225$ ,  $h_1 \div h_0 = \frac{4}{5}$ ,  $h_0 - h_1 = 45$ ; and (311) become

$$y = D_{t} = \frac{2 \times 9000 \times 450^{3}}{25000000 \times 45^{2}} \left\{ 180 - 225 \left( 1 - 2.302585 \log \frac{5}{4} \right) \right\}$$
$$= \frac{9}{125} \times 5.206 = 0.37486 \text{ inch.}$$

For the fourth part,  $EF = \frac{1}{8}I$ , fixed at F;  $h_1 = 30$   $h = h_0 = 285$ ,  $h_1 \div h_0 = \frac{20}{19}$ ,  $h_0 - h_1 = -15$ ;

$$y = D_4 = \frac{9}{125} \left\{ 300 - 285 \left( 1 + 2.302585 \log \frac{20}{19} \right) \right\} = \frac{9}{125} \times 0.36$$
  
= 0.0275 inch.

For the second part, BC,  $h = h_0 = 225$ ,

$$y = D_2 = \frac{9}{135}(h_1 - 225 \times 2.302585 \log h_1 + 225 \times 2.302585 \log 225 - 22)$$

For the third part, CE,  $h = h_0 = 285$ ,

$$y = D_3 = \frac{8}{125}(h_1 - 285 \times 2.302585 \log h_1 + 285 \times 2.302585 \log 285 - 28)$$

And since  $D_t + D_s = D_s + D_{\varphi}$  we find

$$h_1 = 234.761$$
 inches;

that is, the lowest point, C, is now at the distan

10(234.761 - 180) = 547.61 inches from the left end of the beam. Using this value of  $h_1$ , we find  $D_2$  and  $D_3$ , and have finally,

$$D_1 = 0.3749, D_3 = 0.3622,$$

$$D_2 = 0.0148, D_4 = 0.0275,$$
Deflection at  $C = 0.3897$  inch = 0.3897 inch;

an apparently paradoxical result, since, when the same load, W = 200,000 pounds, was at the centre of the girder of uniform strength, and having the same varying height and same length, l = 100 feet, the deflection was only 0.279 inch at the lowest point. The paradox vanishes, however, when we take into account the difference in the length of the component semi-beams for the two cases. Indeed, it may be easily shown that a girder of uniform height and strength, bearing a concentrated load, both ends being fixed, deflects least when that load is at the centre, and the four component semi-beams are of equal length.

Suppose that, in (307), we have, for —

First semi-beam,

$$x = \frac{1}{2}a'$$
, according to article 100;

second semi-beam,

$$x = \frac{1}{2}[l - \frac{1}{2}a' - \frac{1}{2}(l - a')] = \frac{1}{4}l;$$

third semi-beam,

$$= l - \frac{1}{2}a' - \frac{1}{4}l - \frac{1}{2}(l - a') = \frac{1}{4}l;$$

fourth semi-beam,

$$x=\tfrac{1}{2}(l-a').$$

Then, if u is half the sum of the four deflections, that is, if u = the total deflection of the beam, we have

$$u = \frac{B_1}{2Eh} \left[ \left( \frac{1}{2}\alpha' \right)^2 + 2\left( \frac{1}{4}l \right)^2 + \frac{1}{4}(l - \alpha')^2 \right]. \quad (362)$$

Put

$$\frac{du}{da'} = \frac{B_{t}}{4Eh}(2a' - l) = 0.$$
 (363)

Therefore  $a' = \frac{1}{2}l$  renders u a minimum, since 2a' is positive and l constant.

In a similar manner, from (311), may the position of the load be found on the beam of uniform strength and uniformly varying height, the ends being fixed, when it is required to know what position of a given load gives the least deflection.

Example 3. — Continuous uniform load wl = 400,000 pounds upon the same girder, Fig. 102. Since the moments of the external forces are independent of the height, equation (350) applies here, giving for

$$x = 0$$
,  $M_1 = -\frac{3}{82} \times 400000 \times 1200 = -45000000$  inch-pounds;  $x = \frac{1}{2}l$ ,  $M_c = \frac{1}{8} \times 400000 \times 1200 + M_1 = 15000000$  inch-pounds;  $x = l$ ,  $M_2 = M_1$ .

By equations (360) and (361), we find—

At left end, section of top chord,

$$S'' = \frac{45000000}{10000 \times 180 \times 0.99503} = 25.125 \text{ inches.}$$

At left end, section of bottom chord,

$$S' = \frac{45000000}{8000 \times 180 \times 1} = 31.250$$
 inches.

At centre, section of top chord,

$$S' = \frac{15000000}{8000 \times 240 \times 0.99503} = 7.851$$
 inches.

At centre, section of bottom chord,

$$S'' = \frac{15000000}{10000 \times 240 \times 1} = 6.250$$
 inches.

At right end, section of top chord,

$$S'' = \frac{45000000}{10000 \times 300 \times 0.99503} = 15.075 \text{ inches.}$$

At right end, section of bottom chord,

$$S' = \frac{45000000}{8000 \times 300 \times 1} = 18.750$$
 inches.

The deflection must be the same as in example  $\Gamma$ ; viz.,  $D_1 + D_2 = 0.279$  inch, since the centre of gravity of each of the two loads is at the same point, and the unit strain the same.

107. Beam of Uniform Strength and Uniformly Varying Height, fixed at One End, and simply supported at the Other. — Since the position of the point of contrary flexure depends upon the moments due the external forces, which moments are independent of the height of the girder, we already have, in articles 104 and 105, the point of contrary flexure, and the moment at the fixed end,  $M_2$ , for the present case of uniformly varying height, if the load be either concentrated or uniform and continuous.

The cross-section at any point is given generally by equation (332), the deflection of each of the three component semi-beams

by (311), and the equation  $D_1 = D_2 + D_3$  fixes the lowest point.

Example 1. — Take a girder of the same varying height and same length as in the examples of article 106, Fig. 102, of wrought-iron, but now fixed at the right end and simply supported at the left; that is, let E=25,000,000,  $B_1=9,000$ ,  $C_1=8,000$ ,  $T_1=10,000$ , l=1,200 inches,  $h_0=180$  inches = height at left end,  $h_1=300$  inches = height at fixed end,  $\tan\alpha=\frac{25-15}{100}=0.1=\tan$  of slope of top chord,  $\cos\alpha=0.99503$ ,  $\tan\beta=0$ ,  $\cos\beta=1$ , since bottom chord is horizontal. Let the load W=200,000 pounds be at the distance  $a'=\frac{2}{3}l$  from the unfixed end, the point of contrary flexure being at  $x=\frac{1}{3}(l+a')=\frac{5}{3}l$ . Then, from (352),

Moment at fixed end = 
$$M_2$$
 =  $-200000 \frac{3(1-\frac{3}{8})/}{1+\frac{3}{8}}$  =  $-320000000$  inch-pounds.  
Moment at weight,  $M_{\alpha'}$  =  $320000000$  inch-pounds.  
Moment at left end,  $M_1$  = 0.

At the left end, (360) and (361) give the chord cross-sections = 0; but, of course, as before shown and exemplified for all such cases, the end must be enlarged to bear the shearing and crushing strains with permanent safety.

At the load, (360) gives

S' = 15.46 inches = section at top.

At the load, (361) gives

S'' = 12.31 inches = section at bottom.

At fixed end, (361) gives

$$S'' = 10.72$$
 inches = section at top.

At fixed end, (360) gives

$$S' = 13.33$$
 inches = section at bottom.

Applying equation (311) to the three parts of this beam, AB, BD, DE, we find the deflection, Fig. 103,—

AB, fixed at B; 
$$h_0 = 180$$
,

$$y = D_1 = \frac{9}{125}[h_1 - 180 \times 2.302585(\log h_1 - \log 180) - 180].$$

$$BD$$
, fixed at  $B$ ;  $h_0 = 280$ ,

$$y = D_2 = \frac{9}{125}[h_1 - 280 \times 2.302585(\log h_1 - \log 280) - 280].$$

$$DE$$
, fixed at  $E$ ;  $h_0 = 280$ ,  $h_1 = 300$ ,  $h_0 - h_1 = -20$ ,  $\frac{h_1}{h_0} = \frac{15}{14}$ ,

$$y = D_3 = \frac{9}{125}[300 - 280(2.302585 \log \frac{15}{14} + 1)] = \frac{9}{125} \times 0.681$$
  
= 0.049 inch.

From the equation  $D_1 = D_2 + D_3$  we have

$$h_1 = 229.731$$
 inches,

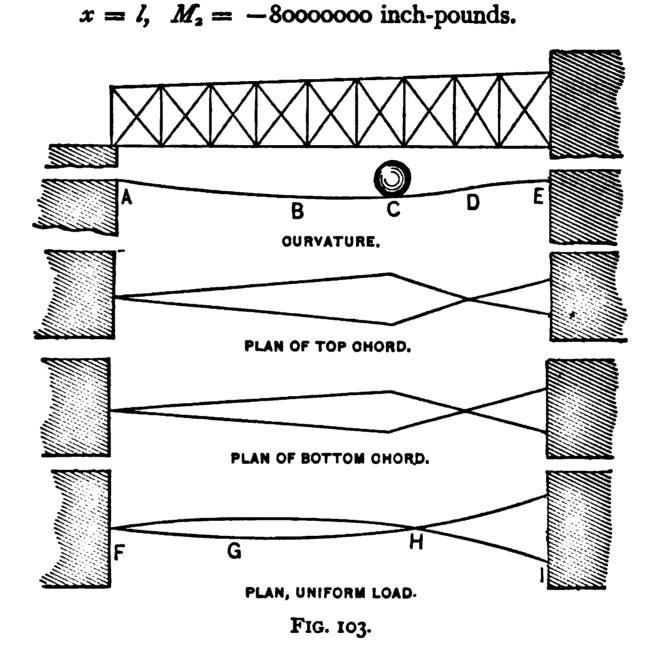
$$\therefore$$
  $D_1 = 0.419$  inch = deflection at lowest point, B.

Example 2. — Take the same girder, with the same conditions, as in example 1, except that the load is now wl = 400,000 pounds, uniformly distributed.

The moments are found by (356); thus,

$$x = 0, M = M_1 = 0.$$

$$x = \frac{1}{8}, M = \frac{400000 \times 1200}{18} = 26666666.$$



Using these moments, we find the required cross-sections, by means of equations (360) and (361), as follows:

 $x = \frac{1}{8}l$ , section of top chord, S' = 15.23 square inches.

 $x = \frac{1}{8}l$ , section of bottom chord, S'' = 12.12 square inches.

x = 1, section of top chord, S'' = 26.80 square inches.

x = l, section of bottom chord, S' = 33.33 square inches.

Deflection for the three parts, by (311):—

FG, fixed at G;  $h_0 = 180$  inches,

$$y = D_1 = \frac{9}{125}[h_1 - 180 \times 2.302585(\log h_1 - \log 180) - 180].$$

GH, fixed at G;  $h_0 = 260$  inches,

$$y = D_2 = \frac{9}{125}[h_1 - 260 \times 2.302585(\log h_1 - \log 260) - 260].$$

HI, fixed at I;  $h_0 = 260$ ,  $h_1 = 300$ ,  $h_0 - h_1 = -40$ ,  $\frac{h_1}{h_0} = \frac{15}{13}$ ,

$$y = D_3 = \frac{9}{125} [300 - 260(2.302585 \log \frac{15}{13} + 1)].$$

From  $D_1 = D_2 + D_3$ , we find  $h_1 = 226.547$  at the point x = 10(226.547 - 180) = 465.47 inches,

 $\therefore D_1 = 0.37 \text{ inch}$ 

at G, Fig. 103.

### SECTION 5.

### Camber.

108. Camber is the slight upward curving or crowning that is sometimes given to a girder, in order to obviate the sagging which would otherwise result from the deflection of the same girder made without this slight arching. The effect of camber is, therefore, to keep the track line straight under the working-load, and thereby prevent that increase of stress which would otherwise be developed by the falling and rising of loads moving rapidly along a line originally straight. In no other respect

does camber augment the efficiency of the structure. Sometimes, however, a greater upward curvature than that here contemplated is given to the floor line of highway bridges, as being more pleasing to the eye; but so large a convexity, if effected in the girder itself, is always at the expense of material or of efficiency, as will appear from a comparison of the capabilities of two girders shaped like Figs. 23 and 80, of equal length and equal height between axes of chords.

It is evident that camber may be given to the floor or track line in three ways:—

- 1st, The girders may be made in normal shape, and the floor or track line be raised sufficiently to counteract the deflection due, the total load. In this case the two chords of each girder will sag, while the cambered floor line becomes straight under load.
- 2d, The chord which carries the floor line may be cambered, while the other is built in normal shape. In this case the uncambered chord will sag, while the other assumes its normal shape under load.
- 3d, The girder may be so built, that, before the load is imposed, its proper floor line will have a deflection equal and opposite to the deflection due the total load, and that the whole girder will assume its normal shape under load.

We need examine and exemplify only the second and third cases.

rog. Change of Length calculated from the Unit Strain.

— If  $\lambda_1 =$  total contraction for the original length  $l_1$ , and  $\lambda_2 =$  total elongation for the original length  $l_2$ , of any strained member, we have, within the elastic limit where the amount of displacement per unit of length varies as the stress,—

For compressed member,

$$\lambda_i = \frac{C_i l_i}{E_c}; \qquad (364)$$

for extended member,

$$\lambda_2 = \frac{T_1 \ell_2}{E_t}; \qquad (365)$$

 $C_1$  and  $T_2$  being the allowed unit strains, and  $E_c$  and  $E_t$  the moduli of compressive and of tensile elasticity respectively.

The total difference between the lengths of the two chords of a girder after deflection is, therefore,

$$\lambda = \lambda_1 + \lambda_2 = \left(\frac{C_1}{E_c} + \frac{T_1}{E_t}\right) l, \qquad (366)$$

provided the chords were of equal length, I, before deflection, and of uniform strength.

If an originally straight girder of equal and parallel chords take the circular form, Fig. 104, after deflection, the neutral line being midway between the chords, we must have for it,

$$\lambda_{i} = \lambda_{2} = \frac{C_{i}l}{E_{c}} = \frac{T_{i}l}{E_{t}} = \frac{\frac{1}{2}(C_{i} + T_{i})l}{\frac{1}{2}(E_{c} + E_{t})} = \frac{B_{i}l}{E},$$
 (367)

$$\therefore \quad \lambda = \lambda_1 + \lambda_2 = \frac{C_1 l}{E_c} + \frac{T_1 l}{E_t} = \frac{2B_1 l}{E}, \quad (368)$$

if  $E = \frac{1}{2}(E_c + E_t) = \text{modulus of transverse elasticity, and}$   $B_i = \frac{1}{2}(C_i + T_i) = \text{bending unit strain.}$ 

110. Elongation and Contraction calculated from Deflection. — Let ABCD, Fig. 104, represent an open-built semi-girder fixed horizontally at A and C, and having its deflection, D = NH, greatly exaggerated in the figure; the actual lines NFM and HM being sensibly equal each to l, the original

length of the parallel chords AB and CD of the semigirder whose height is AC = k.

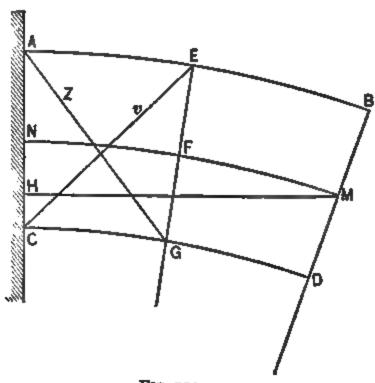


FIG. 104.

We may without appreciable error, for the present purpose, regard the deflection curve as circular.

Take, as radii of curvature, r for neutral line NM,  $r + \frac{1}{2}h$  for the extended chord,  $r - \frac{1}{2}h$  for the contracted chord. Then, from the geometry of the figure, we have

$$P = D(2r - D),$$

$$\therefore r = \frac{P}{2D} + \frac{1}{2}D.$$

$$r = \frac{P}{2D}, \qquad (369)$$

since D is very small compared with r.

Also, from the figure,

$$\frac{r}{r + \frac{1}{2}h} = \frac{l}{l + \lambda_1},$$

$$\therefore l + \lambda_1 = \frac{l(r + \frac{1}{2}h)}{r}, \qquad (370)$$

which is the length to be given to the chord in compression, the figure being inverted. Again,

$$\frac{r}{r - \frac{1}{2}h} = \frac{l}{l - \lambda_2},$$

$$\therefore l - \lambda_2 = \frac{l(r - \frac{1}{2}h)}{r}, \qquad (371)$$

the length required for the chord in tension, the figure being inverted.

Subtracting (371) from (370), we find the total difference in length required,

$$\lambda = \lambda_1 + \lambda_2 = \frac{hl}{r} = \frac{2Dh}{l}, \qquad (372)$$

after eliminating r by means of (369).

It may be noted that (368) and (372) give us  $D = \frac{B_1 l^2}{Eh}$ , which is equation (308) for the semi-girder of the length l.

In (369) and (372), we must, of course, put  $\frac{1}{2}l$  for l, when we apply these equations to a girder of the length l supported at both ends.

straight, horizontal bottom chord, to which the moving-load is to be applied. This supposition includes Classes II., IV., VII., IX., X., and XII. of article 49.

We may, by the formula proper for the given girder, find the deflection at each panel point, or apex, of the bottom chord. If we now assume that no apices of the bottom chord are to be moved horizontally, by reason of the adopted camber, we must theoretically *increase* each normal panel length, c, of this chord, in the ratio  $\frac{c + \Delta c}{c} = \frac{1}{\cos \beta}$ ;  $\beta$  being the inclination to the horizon of any panel length,  $c + \Delta c$  of the bottom chord when cambered, and  $\Delta c$  being the change of length in the bottom chord for any panel, by reason of the adopted camber. Also,  $\tan \beta = \frac{D - D_{\rm I}}{c}$ ,  $D_{\rm I}$  and D being the deflections at any two consecutive apices.

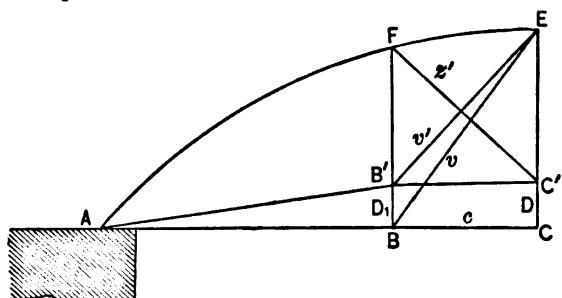


FIG. 105.

Then each vertical member, as FB, Fig. 105, coming down to a lower apex, B, must be shortened by the amount of deflection,  $D_1$ , computed for the apex; and each diagonal or brace, BE = v, terminating at the same apex, must be shortened in the ratio

$$\frac{v'}{v} = \left\{ \frac{c^2 + (h - D_1)^2}{c^2 + h^2} \right\}^{\frac{1}{4}},\tag{373}$$

where h is the normal difference of level between the ends of any diagonal.

In this case the effective depth of the girder at the centre has been lessened by D.

Example. — Let us find the changes of length required to effect camber in the bottom chord alone of a wrought-iron parabolic bowstring, where l = 2,400 inches, h = 225, n = 16 panels of  $c = \frac{2400}{16} = 150$  inches each,  $C_1 = 6,698$  pounds,  $T_1 = 10,000$  pounds,  $B_1 = 8,349$  pounds, and E = 24,000,000 after the frame has taken its permanent "set;" but, as explained in article 90, we will, for the present purpose, take E = 16,000,000, to provide for any sagging that might otherwise be caused by the first full load, beyond what the elasticity of the frame can recover.

By equation (319), putting  $\frac{1}{2}l = 1,200$  for l, we have deflection at centre,

$$D = \frac{1.386295 \times 8349 \times 1200^{2}}{16000000 \times 225} = 4.6297 \text{ inches.}$$

Now we may, by using equation (318), find the deflection at each panel point; but it will be practically accurate, and more simple, to regard the cambered bottom chord as a parabola, having the central height D=4.6297 inches, and then find, by equations (136) and (137), both the normal heights, h, and the height of each lower apex after camber is effected. Thus, (136) now becomes

$$y = \left(1 - \frac{4x^2}{2400^2}\right) \times 4.6297$$

for bottom chord, and

$$y = \left(1 - \frac{4x^2}{2400^2}\right) \times 225$$

for top chord; the origin being at the middle of the bottom

chord in its normal shape. From these equations and (373) we compute D, v', z', h, in inches.

x	À	D	ΔD	k - D	$k_r - D_{r+1}$	$k_{r+1} - D_r$	4	g!
0	225.00	4.63	-0.07	220.37	220.44	216.85	266.64	263.67
150	221.48	4.56	-0.22	216.92	217.14	206.38	263.91	258.14
300	210.94	4.34	<b>—0.36</b>	206.60	206.96	189.09	255.60	241.36
450	193.43	3.98	-0.51	189.45	189.96	164.77	242.04	222.82
600	168.75	3.47	-0.65	165.28	165.93	133.64	223.68	200.90
750	137.11	2.82	-0.79	134.29	135.08	95.62	201.86	177.88
900	98.44	2.03	-0.95	96.41	97.36	50.71	174.76	1 58.34
1050	52.74	1.08	-1.08	51.66	52.74	-	_	-

Theoretically, the end panel lengths of bottom chord, where the inclination,  $\beta$ , is greatest, would become

$$(150^2 + 1.085^2)^{\frac{1}{2}} = 150.0039$$
 inches.

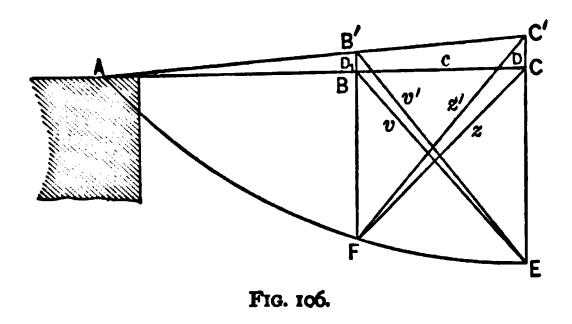
But this is practically equal to the normal length, 150 inches; hence we will not change the panel lengths of the bottom chord.

It will be perceived that the girder thus cambered becomes the parabolic crescent.

Instead of computing dimensions as above, it is evident that the elevation may be drawn accurately, on a large scale, from the central deflection D, and h and l; and then all desired lengths can be taken off as accurately as the work will be "laid out" in the shop. The camber curve may always be drawn circular for an originally straight chord.

112. Similarly, if we would camber only the straight upper horizontal chord of Classes III., IV., VIII., IX., XI., XII., of

article 49, without moving appreciably the upper apices horizontally, we must increase the normal length of each vertical



member by the deflection due at its place, and the normal length of each diagonal in the ratio

$$\frac{v'}{v} = \left\{ \frac{c^2 + (h+D)^2}{c^2 + h^2} \right\}^{\frac{1}{2}}.$$

See Fig. 106, where the efficient depth of the girder has been increased at the centre by the value of the deflection D.

Example. — Let us invert the uncambered girder of article 111, and effect the same amount of camber, D=4.6297 inches, in the straight top chord alone. We have the same values of D and h as before, and readily find the required lengths of verticals and diagonals, in inches, numbering from the centre.

$$v' = \left[c^2 + (h_r + D_{r+1})^2\right]^{\frac{1}{2}} = c \left\{ 1 + \left(\frac{h_r + D_{r+1}}{c}\right)^2 \right\}^{\frac{1}{2}}, \quad (374)$$

$$s' = \left[c^2 + (h_{r+1} + D_r)^2\right]^{\frac{1}{2}} = c \left\{1 + \left(\frac{h_{r+1} + D_r}{c}\right)^2\right\}^{\frac{1}{2}}.$$
 (375)

£	À	D	À + D	$k_r + D_{r+1}$	•	$k_{r+1} + D_r$	z.º
0	225.00	4.63	229.63	229.56	274.22	226.11	271.34
150	221.48	4.56	226.04	225.82	271.09	215.50	262.57
300	210.94	4.34	21 5.28	214.92	262.08	197.77	248.22
450	193.43	3.98	197.41	196.90	247.53	172.73	228.77
600	168.75	3-47	172.22	171.57	227.90	140.58	205.58
750	137.11	2.82	139.93	139.14	204.59	101.26	180.98
900	98.44	2.03	100.47	99.52	180.01	54-77	159.69
1050	52.74	1.08	53.82	-	-	-	_

In like manner may we effect camber in a straight chord of any one of the classes cited in this and the preceding article. And, if it is required to preserve the normal height between chords after camber, we must change both.

113. When it is desired that the effective depth of the girder be not altered by the camber, then both chords must be displaced vertically by the amount of the deflection at the several apices, and in the opposite direction.

In articles III and II2 we have made no appreciable change in the length of either chord by reason of camber; and, of course, the length of each chord will be changed as the load takes out the camber.

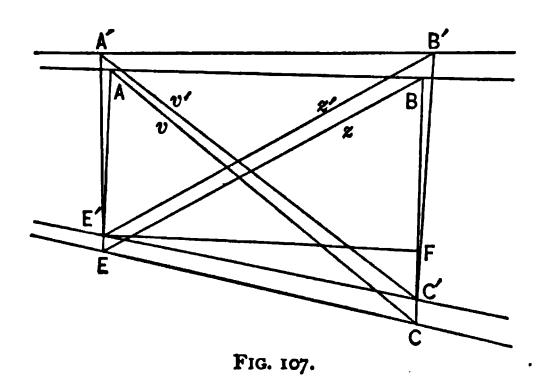
Strictly, regarding camber as the inverse of the operation performed by deflection, we should increase the normal length of the compressed chord by  $\lambda_1$ , equation (364), and diminish that of the stretched chord by  $\lambda_2$ , equation (365); but, since the whole change of length required in either chord is very small for each panel, we shall distribute, in the present case, the whole difference,  $\lambda = \lambda_1 + \lambda_2$ , of length due to deflection additively among the panel lengths of the compressed chord. Hence camber thus produced will require no change in the normal panel lengths of the chord in tension, which, under the load, will resume its normal line, but increased in length by  $\lambda_2$ .

At the same time, the compressed chord loses  $\lambda_i$  of its increment, and retains  $\lambda_i$ .

This change of length in either chord which rests upon the points of support, may, if necessary, be provided for in the same manner that provision is made for the effect of change of temperature. The length of any vertical member is not to be altered appreciably for camber in this case, since the vertical displacement of each chord is assumed to be the same for any given value of x; and the slight change in their length caused by the spreading of the verticals to fit the change in the compressed chord, is hardly measurable.

But the length of any diagonal member will be changed as follows:—

Let ABCE, Fig. 107, represent any one of the n normal panels of a girder, and A'B'C'E' the same panel when cambered



by adding  $\frac{\lambda}{n}$  to  $\frac{l}{n} = c$ , the horizontal projection of the chord AB in compression. The panel points in both chords are displaced vertically by the deflections  $D_r = CC' = BB'$ , and  $D_{r+1} = EE' = AA'$ , appreciably; and the points A and B

are removed horizontally by the space  $\frac{\lambda}{n}$ . Hence practically we have

$$E'C' = EC,$$

$$A'B' = AB + \frac{\lambda}{n},$$

$$A'E' = A\hat{E},$$

$$B'C' = BC,$$

$$E'F = \frac{l}{n}, \quad FC' = \frac{l}{n}\tan\beta;$$

 $\beta$  being now the inclination of EC or E'C' to the horizon.

$$\mathbf{z}' = E'B' = \left\{ \left( \frac{l}{n} + \frac{\lambda}{2n} \right)^2 + \left( h_r - \frac{l}{n} \tan \beta \right)^2 \right\}^{\frac{1}{2}}, \quad (376)$$

$$v' = A'C' = \left\{ \left( \frac{l}{n} + \frac{\lambda}{2n} \right)^2 + \left( h_{r+1} + \frac{l}{n} \tan \beta \right)^2 \right\}^{\frac{1}{4}}.$$
 (377)

Instead of  $\frac{\lambda}{n}$ , we may evidently employ equations (364) and (365) in finding the proper increment for any panel length of compressed chord.

EXAMPLE. — Take  $BC = h_r = 240$  inches,  $AE = h_{r+1} = 200$  inches,  $E'F = \frac{l}{n} = c = 150$  inches,  $\frac{\lambda}{n} = \frac{\lambda_1 + \lambda_2}{n} = 0.12$  inch,  $\frac{\lambda}{2n} = 0.06$  inch,  $\tan \beta = \frac{1}{2} \times \frac{240 - 200}{150} = \frac{2}{15}$ . Then

$$A'B' = AB + 0.12,$$
  
 $z' = [(150.06)^2 + (240 - 150 \times \frac{2}{15})^2]^{\frac{1}{2}} = 266.304 = v'$   
 $= [(150.06)^2 + (200 + 20)^2]^{\frac{1}{2}}$  inches.

# CHAPTER VII.

#### PILLARS.

### Section 1.

## Strength of Pillars, by Rational Formula.

umns, posts, struts, props, braces in compression, and, in a word, every member in a structure whose function it is to resist compressive force applied at its end, and, in general, in the line of the longitudinal axis of the member.

It is assumed that a pillar has no lateral support or pressure applied between its ends, except when, owing to an unavoidable existing lateral force (as, for example, the weight of a horizontal strut), a counter-force is applied as a balance. But a pillar may have its ends in the conditions known as round, hinged, flat, imbedded, fixed; the two ends being in the same or in different conditions. Pillars may be long or short, solid or hollow; may have a uniform or variable cross-section of any desired form.

Long pillars yield chiefly by bending and breaking across; short blocks of ordinary building material yield by being crushed without bending, properly so called. At what exact ratio of length to diameter bending first takes place in a given material, is not at present very definitely ascertained; but it will be safe to assume, in the present state of our knowledge, that bending will occur when this ratio is as low as three for such

material as wrought-iron. Experiment has shown what, perhaps, we might have inferred from a stalk of wheat, — that material is saved by using hollow instead of solid pillars to support a given load.

115. Pillars of Uniform Cross-Section.

Let l = length of pillar,

h = least diameter,

r = least radius of gyration of cross-section,

S = area of cross-section,

D =greatest deflection of pillar;

all in inches.

E =modulus of transverse elasticity,

C =crushing-strength of standard specimen of the material,

P = breaking-weight applied at the end of the pillar and in the line of its axis before deflection,

 $Q = \frac{P}{S}$  = breaking-weight per square inch of cross-section; all in pounds per square inch.

 $J = Sr^2$  = least moment of inertia (so called) of cross-section.

 $M_x$  = moment of forces developed in any normal cross-section by the given load P.

 $M_{\rm r}$  = the end moment at the lower end when that end is fixed.

 $M_2$  = the end moment at the top when the upper end is fixed.

Suppose the pillar vertical, Figs. 108, 109, 110, and take the origin of rectangular co-ordinates at the lowest point of the pillar's axis, which call also the axis of x; that of y being horizontal.

Then, from equations (15), (93), and (187), we have the moment at any height, x,

$$M_x = -EI\frac{d^2y}{dx^2} = \frac{M_1 - M_2}{l}x - M_1 + Py,$$
 (378)

wherein no account is taken of the modified condition of every cross-section due to the longitudinal pressure, Q, per unit.

Now, since the full unit strength of the cross-section of the unloaded pillar is C, and the remaining unit strength of the loaded pillar's cross-section is (C - Q), it follows that the expression for the moment of the internal forces developed in any cross-section must be diminished in the ratio  $\frac{C - Q}{C}$ .

We then have

$$M_x = -Pe^2 \frac{d^2y}{dx^2} = \frac{M_1 - M_2}{l} x - M_1 + Py \qquad (379)$$

if

$$\mathbf{e}^2 = \frac{EI(C-Q)}{PC} = \frac{Er^2(C-Q)}{QC}.$$

There will be three cases, according as we consider neither, both, or one only, of the ends fixed.

CASE I. — If neither end can produce any moment,  $M_1 = M_2 = 0$ ; and we have, from (379),

$$\epsilon^2 \frac{d^2 y}{dx^2} = -y. \tag{380}$$

Multiplying by 2dy,

$$2\varepsilon^2 \frac{dy \, d(dy)}{dx^2} = -2y \, dy.$$

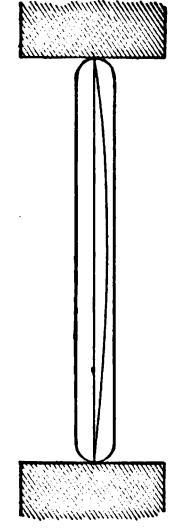
Integrating this equation, and putting  $a^2$  for the arbitrary constant of integration,

$$\varepsilon^2 \frac{dy^2}{dx^2} = a^2 - y^2,$$

from which

$$\frac{dx}{s} = \frac{dy}{(a^2 - y^2)^{\frac{1}{2}}}.$$

Integrating again between the limits, for x, 0 and l; and for y, 0 and 0;



$$\therefore l = \epsilon \left[ \sin^{-1} \frac{y}{a} \right]_{0}^{\circ} = \epsilon n \pi,$$

where *n* may be any whole number; but, in order that *P* may have the least value it can have, consistent with the bending of the pillar necessarily assumed in establishing equations (378) and (379), *n* must be equal to unity. (See Rankine's "Applied Mechanics," p. 352.)

$$\therefore l^{2} = \pi^{2} e^{2} = \frac{\pi^{2} E r^{2} (C - Q)}{QC},$$

$$Q = \frac{C}{1 + \frac{C l^{2}}{\pi^{2} E r^{2}}},$$
(381)

FIG. 108. which is the formula for pillars with rounded ends that can generate no end moments, Fig. 108. The curved line shows the deflected axis.

Case II. — If both ends of the pillar are equally fixed, Fig. 109, so that the elastic curve at each end, after flexure, has for its tangent the original undeflected axis, then, in equation (379),

$$M_{\cdot} = M_{\cdot}$$

whence

$$Pe^2\frac{d^2y}{dx^2}=M_1-Py. (382)$$

Multiplying by 2dy, equation (382) becomes

$$2Pe^{2}\frac{dy\,d(dy)}{dx^{2}}=2M_{1}dy-2Py\,dy.$$

Integrating, we find

$$Pe^{2}\frac{dy^{2}}{dx^{2}}=2M_{1}y-Py^{2}+a, \qquad (383)$$

where a, the arbitrary constant, must vanish, since  $\frac{dy}{dx} = 0$  when y = 0. Hence, from (383),

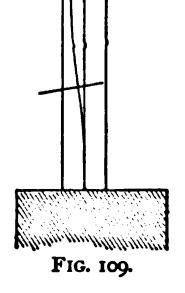
$$\frac{dx}{\varepsilon\sqrt{P}} = \frac{dy}{(2M_1y - Py^2)^{\frac{1}{2}}}.$$

Integrating again, with the condition that y = 0 when x = 0, there results, after cancelling  $\sqrt{P}$  from the denominators,

$$\frac{x}{s} = \sin^{-1}\left(\frac{Py - M_1}{M_1}\right) + \frac{\pi}{2}. \quad (384)$$

Also we have y = 0 when x = 4 so that (384) becomes

$$\frac{1}{8} = \sin^{-1}(-1) + \frac{\pi}{2} = \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi,$$



to be consistent with the permanence of l and with the least positive value of P. Therefore

$$l^{2} = 4\pi^{2} \epsilon^{2} = \frac{4\pi^{2} E r^{2} (C - Q)}{QC},$$

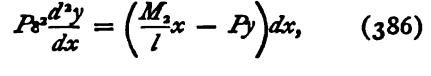
$$Q = \frac{C}{1 + \frac{Cl^{2}}{4\pi^{2} E r^{2}}},$$
(385)

which is the formula for pillars with both ends equally and fully fixed.

Case III. — When only one end of the pillar is fixed, Fig. 110, and the other end can cause no end moment, we

have (say)  $M_1 = 0$ , and derive, from equation





an equation whose second member cannot be integrated "without specific connection between x and y," unless there is such a relation among the quantities which compose it that the member shall be wholly a function of x. (See De Morgan's "Differential and Integral Calculus," p. 208.) Not knowing the specific connection between x and y, nor what relation among its component quantities there may be to render the second member a "function of x only," I shall here assume a connection expressed by the equation of a parabola; viz.,

$$y = x + bx^2, \qquad (387)$$

where, since y = 0 when x = l,  $b = -\frac{1}{l}$ ; and shall attempt only an approximate solution for this third case.

Putting this value of y into equation (386), it becomes

$$Pe^2 \frac{d^2y}{dx^2} = mx - Pbx^2$$

if 
$$m = \frac{M_2 - Pl}{l}$$
.

Fig. 110.

Integrating, with the condition that  $\frac{dy}{dx} = 0$  when x = l, since the upper end of the pillar is now fixed,

$$\therefore Pe^{2}\frac{dy}{dx} = \frac{1}{2}m(x^{2} - l^{2}) - \frac{1}{3}Pb(x^{3} - l^{3}).$$
 (388)

Integrating again between the limits 0 and y, 0 and x,

$$\therefore P \epsilon^2 y = \frac{1}{2} m \left( \frac{x^3}{3} - l^2 x \right) - \frac{1}{8} P b \left( \frac{x^4}{4} - l^3 x \right). \quad (389)$$

But y = 0 when x = l; hence, from (389),

,

$$m = \frac{3}{4}Pbl. \tag{390}$$

If in (388) we make  $\frac{dy}{dx} = 0$ , we may find the value of x which renders y = D a maximum. Dividing by x - l, and, by means of (390), eliminating m and Pb, we derive from (388), when y is a maximum,

$$x = \frac{1}{16}l(1 \pm \sqrt{33}) = 0.42153l,$$

since the negative value is not here admissible.

Taking x = 0.42153l,  $y = x + bx^2$ , and  $b = -\frac{1}{l}$ , we find from (389), after restoring  $s^2$  and m, and reducing,

$$Q = \frac{C}{1 + \frac{Cl^2}{22.511Er^2}} = \frac{C}{1 + \frac{Cl^2}{2.28\pi^2 Er^2}},$$
 (391)

which is an approximate formula for pillars having but one end fixed.

The nearness and sufficiency of this approximation will be examined in article 120.

It should be observed here that equations (381), (385), and (391) are applicable to all pillars that yield by bending, of whatever uniform cross-section, and of whatever material constructed. Examples of the application of these three equations may be found in the tables of article 121.

#### SECTION 2.

Hodgkinson's Empirical Formulæ for the Strength of Cast-Iron and Timber Pillars.

116. The eminent English experimenter Mr. Eaton Hodgkinson deduced, from his experiments upon pillars of cast-iron and pillars of timber, formulæ which have found place in all works on applied mechanics.

Using the notation of article 115, and taking those values of the constants which have been adopted by such writers as Rankine and Humber, Mr. Hodgkinson's formulæ for the ultimate strength of cylindrical cast-iron pillars, where the length of each is not less than thirty times the diameter if the ends are flat, and not less than fifteen times the diameter if the ends are rounded, become,—

Solid cast-iron pillars,

$$P = A \frac{h^{3\delta}}{(\frac{1}{2\pi}l)^{3/2}}; (392)$$

hollow cast-iron pillars,

$$P = A \frac{h^{3.6} - h_1^{3.6}}{(\frac{1}{13}l)^{1.7}}; (393)$$

being the pillar's internal diameter, and A "representing the rength of a pillar I foot long, and I inch in diameter, and sing a constant for a given quality of iron, but ranging in lue, for different irons, from 75,000 to 112,000." The mean lues of A adopted by Professor Rankine are,—

Solid pillars with rounded ends,

$$A = 14.9 \text{ tons} = 33376 \text{ pounds};$$

lid pillars with flat ends,

$$A = 44.16 \text{ tons} = 98918 \text{ pounds};$$

hollow pillars with rounded ends,

$$A = 13 \text{ tons} = 29120 \text{ pounds};$$

hollow pillars with flat ends,

$$A = 44.3 \text{ tons} = 99232 \text{ pounds.}$$

It hence results experimentally that "fixing" both ends of a pillar, Fig. 109, enables it to support about three times the load which would break it were the ends unfixed, Fig. 108, and incapable of developing moment. For a pillar fixed at one end and rounded at the other, Fig. 110, Mr. Hodgkinson found the strength to be a mean between the two strengths of the same pillar when both ends are rounded and when both ends are flat. We then have, for cast-iron pillars,—

Solid, one flat and one round end,

$$A = 66147$$
 pounds;

hollow, one flat and one round end,

$$A = 64176$$
 pounds.

When the length is less than 30 or 15 times the diameter respectively, Mr. Hodgkinson first finds P by equations (392) and (393), and then corrects P by means of this supplementary formula;  $P_{I}$  being the corrected value sought.

$$P_{1} = \frac{PCS}{P + \frac{3CS}{4}} = \frac{CS}{1 + \frac{3CS}{4P}},$$
 (394)

$$\therefore Q_{1} = \frac{P_{1}}{S} = \frac{C}{1 + \frac{3CS}{4P}} = \frac{C}{1 + \frac{3C}{4Q}}, \quad (395)$$

which is an empirical equation identical in form with (381), (385), and (391), analytically established.

117. The Hodgkinson formula for the *ultimate* resistance of pillars of *oak* and of *red pine* to crushing by bending, as adopted by Professor Rankine, "Applied Mechanics," p. 365, is, with our notation, article 115,

$$Q = \frac{P}{S} = 500 C \frac{h^2}{I^2}, \tag{396}$$

a formula to be used only when Q < C, the crushing-strength of the material, Table II., article 60.

Applications of the Hodgkinson formulæ are given in tables of article 121.

# SECTION 3.

Gordon's Empirical Formula, with Rankine's Modification.

118. We have, in article 115,  $Q = \frac{P}{S}$  = the direct unit pressure of the load upon every cross-section of the pillar.

Now, if  $B_r$  is the *additional* unit pressure due to bending-moment upon those fibres where the bending-moment is greatest, and if f denote the greatest intensity of unit pressure, we have

$$f = Q + B_{\rm r}. \tag{397}$$

Regarding, with reference to the central moment, a loaded pillar of uniform cross-section as in the condition of a beam supported at both ends, and carrying the central weight  $W = \frac{4PD}{l}$ , since equations (15), (46), and (187) give us

$$M = PD = \frac{1}{4}Wl = \frac{2B_1I}{h},$$
 (398)

we find

$$W1 = \frac{8B_1I}{h} = \frac{48EID}{l^2},$$

from (211);

$$\therefore D = \frac{B_1 l^2}{6Eh}.$$

From which, for a given value of  $\frac{B_i}{E}$ ,

$$D \sim \frac{l^2}{h}.$$

But (398) gives

$$B_{\rm r} \sim \frac{PDh}{I} \sim \frac{PD}{Sh} \tag{399}$$

if  $I = kh^2S$ , k being a constant depending upon the form of the pillar's cross-section (see Table III., article 62);

$$\therefore B_1 \sim \frac{Pl^2}{Sh^2}.$$

Whence (397) becomes

$$f = \frac{P}{S} \left( 1 + \frac{l^2}{ah^2} \right),$$

f and a being constants to be determined by experiment;

$$\therefore \frac{P}{S} = \frac{f}{1 + \frac{l^2}{ah^2}},\tag{400}$$

which is of the form "proposed by Tredgold," and is now known as the "Gordon Formula," having been, after some "disuse, revived by Mr. Lewis Gordon, who determined the values" of a and f, for certain materials, from the results of Mr. Hodgkinson's experiments.

119. If, in equation (399), we put  $Sr^2$  for I, using still the notation of article 115, we find

$$B_1 \sim \frac{PDh}{Sr^2} \sim \frac{Pl^2}{Sr^2}.$$
 (401)

Therefore, from (397),

$$f=\frac{P}{S}\left(1+\frac{l^2}{a_1r^2}\right),$$

$$\frac{P}{S} = \frac{f}{1 + \frac{l^2}{a_1 r^2}},\tag{402}$$

which is Professor Rankine's modification of the Gordon formula; r being the least radius of gyration of the cross-section.

The Gordon (400) and the Rankine (402) formulæ are identical if we make

$$\frac{a_1}{a} = \frac{h^2}{r^2}. (403)$$

120. Supposing f to be constant for varying conditions of the pillar, both a and a, will be found to require different coefficients, according as the pillar has neither, one, or both, of its ends fixed.

Assuming that equations (400) and (402) apply to a pillar fixed in direction at both ends, Fig. 109, we see that the length, *l*, between the points of contrary flexure, is in the condition of a pillar not fixed at its ends, and has only the strength of a pillar of twice its length, 2*l*, fixed at both ends; that is, for a pillar rounded at both ends, we have,—

Gordon's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{4l^2}{ah^2}};$$
 (404)

Rankine's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{4l^2}{a_1 r^2}}.$$
 (405)

Similarly, in Fig. 109, the length, l, between either point of contrary flexure and the remoter end is in the condition of a pillar with one fixed and one rounded end, and has only the strength of a pillar  $\frac{4}{3}l$  in length. We have, then, for a pillar fixed at one end and rounded at the other,—

Gordon's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{16}{9}l^2};$$
 (406)

Rankine's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{16l^2}{a_1 r^2}} \tag{407}$$

This is Mr. Hodgkinson's ingenious explanation of the variation among the strengths of these three classes of pillars, a variation which he discovered by a comparison of the results of his experiments.

If we invert the three numerical co-efficients of the fractions in the denominators of (400), (404), (406), or (402), (405), (407) (viz., I, 4,  $\frac{16}{9}$ ), and multiply the inverted numbers by 4, we have the relation, 4, I, 2.25; while 4, I, 2.28, is the relation of the corresponding constants in equations (385), (381), (391), determined analytically. We may hence infer that the degree of approximation in (391) is close to the true value. Especially, since we can seldom tell the exact amount of influence which given end bearings exert, may we regard (391) practically correct.

TABLE IV.

Values of f and a of the Gordon, and of  $f_i$  and  $a_i$  of the Rankine Formula.

A., American Bridge Company, Chicago, Ill.; K., Keystone Bridge Company, Pittsburgh, Penn.

Material,	Form of	Experi-	Authority.	Gordon Formula.		Rankine Formula		
	Section.	menters.		£		11	a <sub>1</sub>	
Iron, Cast	0	Hodgkinson.	Cordon.	80000	400	-	-	
Iron, Cast		Hodgkinson.	Gordon.	80000	<b>867</b>	-	-	
Iron, Cast .	₽	Hodgkinson,	Gordon.	Booco	133	-	-	
Iron, Wrought.		Hodgkinson.	Gordon.	36000	3000	- 1	-	
Iron, Wrought.	66	Hodgkinson.	Rankine.	36000	3000	36000	36000	
Iron, Wrought .	66	Hodgkinson,	Stoney.	35840	3000	-	-	
Iron, Wrought .		Hodgkinson.	Stoney.	30660	3000	-	-	
Iron, Wrought.		Hodgkinson.	Stoney.	40038	200	-	-	
Iron, Wrought.	LTC+	Davies.	Unwin.	42560	900	-	-	
fron, Wrought.		A, K,	Lovett.	49580	3000	43980	36000	
.ron, Wrought .		A. K.	Lovett,	43725	[COM	3 <b>869</b> 0	36000	
iron, Wrought .	مینیت محکی	- 16	Lovett.	38271	3000	37029	36000	
.ron, Wrought		K.	Lowett.	36403	jacon	33531	36000	
iteel, Mild	0	-	Baker.	67900	1400	-	-	
Steel, Strong	"	-	Baker.	114340	900	-	-	
iteel, Mild		-	Baker.	67200	2480	-	-	
Steel, Strong	46	-	4 .	T14140	2600	-	-	
l'imber	44	Hodgkinson.	Rankine.	7200	#50	-	-	
Dak and Fir .	66	Rondelet,	Stoney.	1.5 C of Table II.	250	_	_	
Stone and Brick,	"	-	Rankine.	C of Table II.	600	_	_	
		<u> </u>	<u> </u>	<u> </u>				

# SECTION 4.

Strength of Pillars computed by the Preceding Formulæ, and compared with the Strength experimentally determined.

rat. The following tables, V., VI., VII., VIII., IX., X., XII., contain data derived from experiments on the strength of pillars, probably as trustworthy as any yet made and published. To these tests the appropriate formulæ, either direct or inverted, have been applied; and the values of f, C, or Q for a given pillar, computed by different formulæ, have been tabulated in the same horizontal line.

In Table XI. no experimental values are given, but the assumed values of E and C are within the limits fixed by experiments upon steel. In Table VII., when the thickness, t, of the metal is less than a fifty-fifth part of the least diameter, h, of the pillar, the computed value of Q, the breaking-weight, in pounds, per square inch, has been diminished in the ratio  $\frac{55t}{h}$ , as seems to be required by the tests.

For columns having rounded or hinged ends, in Table V., the formulæ for those having one flat and one round end have been used, as more in harmony with the tests than the formulæ for columns having no end moments.

It must be confessed that there are anomalies of considerable magnitude in the experiments themselves; and, of course, there appear corresponding variations from the test values in the numbers computed according to the laws of the applied formulæ.

It is to be regretted that we have not, accompanying these tests for Q, also experimental determinations of C and of E, for each pillar tabulated, but have been obliged to use probable mean values of C in all the calculations of Q, and probable mean values of E in all but Table V.

1-007	10s. to un. 60141 60141 43390 43390 40548 40548 40548 4054 40514 33890 3370 3370 3370 41930 41930 41930 41930 41930 41930	(391) 1	
Kabeine, $Q\left\{1+\frac{l^3}{36000t^2}\right\}$	10 to in. 49400 47000 47000 47000 39900 37	2 1+ 1/2 1/2 1/2 1/2 1/2 1/2 1/2 1/2 1/2 1/2	
Cordon, $Q\left\{1+\frac{2^{3}}{3000k^{3}}\right\}$	104. to in. \$4500 \$4500 47400 47700 37500 37500 37500 37500 37500 37500 37500 37500 37500 4100 37100 4100 37100 41	47700 44700 44700 41700 40190	
Breaking. Weight, by Experiment.	lbs. to in. 36600 34600 34600 31500 81700 81700 81700 81700 31500 32000 32000 33000 33000 33000 33000 33000 33000 33000 33000 33000 33000 33000 33000 33000 33000 33000 33000 33000 33000 330000 330000 330000 330000 330000 33000 3	espans segment	
Modulus of Transverse Elasticity,	14	728737F	
Square of Least Radius of Gyration.	20.00000000000000000000000000000000000	5.479 6.733 10.038 8.733 8.935 10.954 11.000	
Length.	ESSERVE STREET THE SECOND STREET	4 4 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6	
Least Diameter.	######################################	8.000 10.000 10.750 10.000 10.000	
Area of Cross- section.	######################################	14 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9	
of Ends.		Hinged Hinged Hinged Hinged Round Hinged Hinged	
	♥♥♥♥★★★¥¥¥¥¥¥¥¥¥¥	4444でNo. No. Selection Miles	Į
<b>!</b>	·····································	2126245	

Mean value of E, equal to 27,311,111, used for Nos. 9 and 19.

### TABLE VI.

### SOLID RECTANGULAR PILLARS OF WROUGHT-IRON.

Flat ends well bedded; Hodgkinson's Experiments.

DATA FROM BINDON B. STONEY'S THEORY OF STRAINS IN GIRDERS AND SIMILAR STRUCTURES.

$k^2=12t^2.$	Assume	E =	27,311,111,	and	C	=	50,000.
--------------	--------	-----	-------------	-----	---	---	---------

	S	À	2	l÷ħ	$Q = \frac{P}{S}$	Excess o	ver Q by
No.	Sectional Area.	Least Diame- ter.	Length.	Ratio of Length to Least Diameter.	Breaking- Weight, by Experiment.	Gordon Formula, $Q = \frac{35840}{1 + \frac{l^2}{3000 k^2}}$	Equation (385), $Q = \frac{C}{I + \frac{Cl^2}{4\pi^2 E r^2}}$
	sq. ins.	ins.	ins.		lbs. per sq. in.	lbs. per sq. in.	lbs. per sq. in.
30	1.0465	1.0230	7.5	7.33 <sup>1</sup>	48682	— <b>13473</b>	_ 134
31	1.0465	1.0230	15.0	14.663	34554	<b>— 1105</b>	+ 10103
32	1.0475	1.0230	30.0	29.326	25327	+ 2527	+ 8489
33	2.9880	0.9960	30.0	30.121	29655	<b>— 2137</b>	+ 3570
34	2,2970	0. <i>7</i> 630	30.0	39.319	27767	- 4115	890
35	1.0486	1.0240	60.0	58.594	17268	<b>–</b> 555	88
36	4.5900	1.5300	90.0	58.524	19987	<b>—</b> 3343	<b> 289</b> 6
37	1.5011	0.5026	30.0	59.689	16853	- 470	- 90
38	5.8166	0.9960	60.0	60.241	17698	- 1479	<b>— 1138</b>
39	2.9950	0.9950	60.0	60.303	18067	<b>– 1865</b>	<b>— 1530</b>
40	2.3090	0.7670	60.0	78. <b>227</b>	12969	<b>— 1179</b>	<b>– 1619</b>
41	4.5300	1,5100	120.0	79.470	10165	+ 1377	+ 914
42	1.0490	1.0240	90.0	87.891	9753	+ 273	- 317
43	2.9915	0.9955	90.0	90.407	9912	<b>_ 289</b>	<b>– 900</b>
44	5.8307	0.9950	90.0	90.452	9280	+ 336	<b>– 27</b> 6
45	1.4980	0.5070	60.0	118.343	5653	+ 670	+ 33
46	1.5110	0.5070	60.0	118.343	5604	+ 719	+ 82
47	2.9750	0.9950	120.0	120.603	4280	+ 1848	+ 1218
48	2.3060	0.7660	120 0	156.658	3379	+ 525	+ 32
49	1.4980	0.5023	90.0	179.176	2410	+ 653	+ 240
50	1.4980	0.5030	120.0	238.659	816	+ 978	+ 714

The substance of section 1 of this chapter, together with some of these tables, appeared first in a contribution by the author to Van Nostrand's "Eclectic Engineering Magazine" for December, 1879, New York.

It is confidently expected that the new United-States Government testing-machine, already in use at the Arsenal in Watertown, Mass., will contribute a set of values for the constants to be used in tension, compression, cross-breaking, and torsion, and for pillars, much more in agreement with the capabilities of the actual members of structures than any values of these constants (if, indeed, they shall turn out to be constants at all) hitherto determined.

# OF WROUGHT-IRON. TABLE VII. - RECTANGULAR TUBULAR PILLARS

DATA FROM BINDON B. STONEY'S "THEORY OF STRAINS IN GIRDERS AND SIMILAR STRUCTURES." Ends flat and well bedded. Hodgkinson's Experiments.

Square of Least Radius of Gyration =  $r^2 = \frac{h^2(h+3b)}{12(h+b)} = \frac{h^2}{6}$  when h=b.

	S	9	<b>*</b>	4+1	1+4	1+1	55¢ ÷ Å	Q = Breaki	= Breaking-Weight, in Lbs., per Sq. Inch,	6., per Sq. Inc
ğ	Sectional Area.	Breadth.	Least Diameter.	Ratio of & to to Thickness of Metal.	Ratio of Length to Least Diameter.	Ratio of Length to Least Radius of Gyration.	Multiplier when $\frac{k}{t} > 55$ .	By Experiment.	Gordon Formula, $Q = \frac{30660}{1 + \frac{l^2}{3000k^2}}$	Equation (385), $Q = \frac{C}{1 + \frac{Cl^2}{4\pi^8 E r^2}}$
	8d. ins.	ins.	ins.							
K	0.50	4.10	4.10	136.6	20.26	71.672	0.40264	10980	9496	<b>16101</b>
25	1.020	4.10	4·10	o.89	29.36	71.672	0.80882	19260	19293	<b>3</b> 0394
<b>2</b> 3	1.484	4.25	4.25	51.0	28.23	69.149	1	25171	24225	25426
ょ	2.395	4.25	4.25	31.7	28.23	69.149	•	21585	24225	25426
55	1.885	8.10	4.10	70.0	29.26	66.452	0.78571	23169	18742	20152
ş	1.532	8.17	4.10	100.6	92.62	66.394	0.54673	15201	13041	14024
57	6.890	8.40	4.25	32.3	28.23	64.104	1	18608	\$4225	25835
<b>38</b>	2.326	8. 8.	4.75	25.1	25.30	27.906	1	86913	25269	<b>26</b> 312
50	8.347	æ. Gr.	4-75	19.0	25.30	26.60I	1	25000	25269	26408
8	2.395	4.25	4.25	31.7	21.10	51.684	1	2320I	26698	26757
61	1.532	8.17	4.10	100.6	22.40	50.002	0.54672	16215	14361	14657
<b>6</b>	1.885	0.10	4.10	0.00	22.30	50.669	0.79710	24111	<b>1</b> 960 <b>2</b>	21384
63	3.550	% 01.0	8.10	63.6	14.94	36.598	0.86478	19732	24678	23930
đ.	2.070	% 0.10	8.10	135.0	14.80	36.252	0.40741	13276	11911	11283
, 65	0.50	4.10	4.10	136.6	14.60	35.762	0.40264	11513	11526	11159
8.	3.550	8.10	8.10	63.6	14.57	35.689	0.86478	23100	24762	<b>3</b> 397 <b>3</b>
8	4.926	8.37	8.37	0.0	14-33	35.101	29916.0	<b>3</b> 03 <b>04</b>	<b>36304</b>	\$5437
8	7.737	8.37	8.37	38.2	14.33	35.101	1	25716	38696	27750
8	8.466	8.40	8.40 0+.8	35.7	r4.33	35.101	ı	26675	38696	2775 <sup>0</sup>
ደ	1.520	4.25	4.25	50.0	14.10	34.538	,	23584	28755	87777
71	2.070	8.10 01.8	8.10 0.10	135.0	11.30	27.679	0.40741	13301	18611	1142
2	1.885	4.10	4.10	68.3	10.70	24.366	0.80527	21889	23782	82718
73	3.550	• 8.1o	8.10	63.6	7.40	18.126	0.86478	23208	99498	24567
74	0.50	4.10	4.10	136.6	7.30	17.881	0.40264	12403	12129	1141
75	1.520	4.25	4.25	50.0	2.8	17.r46	1	27417	29105	28434
ዾ	1.532	8.17	4.10	9.001	6.80	15-493	0.54679	15921	16908	15569
11	20.000	18.00	18.00	36.0	5.33	13.056	1	30101	69 <b>1</b> 0E	28528
82	1.885	8.10 01	4:10	68.3	2.4	10.798	0.80597	26010	60678	Sook &
Ç		¢.	α -	- Y'Y		000	97.70	(1)	7-7-	•

TABLE VIII.

HOLLOW CYLINDRICAL PILLARS OF WROUGHT-IRON.

Ends flat and well bedded. Hodgkinson's Experiments.

Data from Bindon B. Stoney's "Theory of Strains in Girders and Similar Structures."  $\hbar^2 = 8r^2$ .

	S	k	1 + h	1+4	h + 4	Q = Brea	king-Wt., in Lb	s., per Sq. Inch,
No.	Sectional Area.	Diameter.	Ratio of Length to Diameter.	Ratio of Length to Radius of Gyration.	Ratio of Diame- ter to Thick- ness of Metal.	By Experi- ment,	Gordon Formula, $Q = \frac{40050}{1 + \frac{l^2}{3000k^2}}$	Equation (385), $Q = \frac{42290}{1 + \frac{Cl^2}{4\pi^2 K r^2}}$
	sq. ins.	ins.						
80	0.444	1.495	80,00	226,274	15.00	14673	12782	12873
81	0.610	1.964	60.00	172.816	18.80	23206	18204	18127
82	1.435	2.340	51.28	145.042	10,80	22179	21342 ,	21810
83	1.605	2.350	51.00	144.250	9.70	21572	21451	21926
84	0.804	2.490	47.80	135.199	23.27	29798	92735	23289
8 <sub>5</sub>	0.444	1.495	40.00	113.137	15.00	31180	<b>2</b> 0130	26914 26959
87	1.350 0.610	3. <b>000</b> 1.964	40.00 30.50	86, <b>2</b> 67	20,00	<b>27</b> 671 33299	30571	31745
88	1.414	3.035	29.60	83.721	18.00	29789	30997	32212
89	1.707	4.050	29.60	83.721	29.00	27657	30997	32212
90	1.900	4.060	29.60	83.721	26.10	26263	30997	32212
91	1.371	2.335	25.70	72.690	11.40	29998	32823	34219
92	1.472	2.350	25.50	72,125	10.60	29330	33024	34321
93	0.804	2.490	24.10	68.165	23.27	35100	33554	35025
94	1.613	4.052	22,20	62.791	30.90	33331	34399	3596z
95	2.879	4.000	22,20	62 <b>.79</b> 1	16.50	<b>2</b> 6046	34399	35961
96	2.897	4.000	22,20	62.791	16.00	26503	34399	35961
97	2.837	4.000	22.20	62.791	16.50	27816	34399	35961
98	0.804	2.490	21.00	59.397	23.27	36489	34917	36536
99	0.444	1.495	20,00	56.569	15.00	34220	35338	37004
100	1.800	6.180	19.40	54:871	65.00	33375	35586	37280
101	2.540 0.610	6.360 1.964	18.90	53.334	49.00 18.80	3598 <b>5</b> <b>3698</b> 0	35789	37524 39028
103	2.895	3.995	15.30 15.00	43.275 42.426	16.30	30024	37151 37256	39145
104	2.848	3.995	15.00	42,426	16.50	34453	37256	39145
105	2.547	6.366	14.10	39.881	48.90	41664	37561	3948 <b>7</b>
106	1.407	2.343	12.80	36.204	11.10	38214	37976	39953
107	1.435	2.335	12.80	36.204	11.40	36639	37976	39953
108	1.435	2.335	12.80	36.204	11.40	35389	37976	39953
109	1.651	2.383	12.50	35-355	9.70	33107	38067	40055
110	1.358	2.343	12.30	34.790	11.60	39569	38127	40030
111	1.554	2.373	12.20	34.507	10.27	36906	38157	40155
112	1.799	6.175	9.70	28.075	61.10	38355	38832	40853
113	1.414	3.000	9.30	26.305	19.60	37392	38928	41023
114	2.845	4.000	7.00	19.799	16.00	47844	39406	41182
115	2.850	4.026	6.95	19.657	16.00	48567	39415	41573
116	I 799	6.125	4.90	13.859	62.50	41361	39732	41931
<b></b>		<del></del>	<del>`</del>		<del> </del>	<del></del>		

Assume E = 24,000,000. C = 42,290, f = 40,050, Means.

### TABLE IX.

# SOLID CYLINDRICAL PILLARS OF CAST-IRON.

Ends flat and well bedded. Hodgkinson's Experiments.

Data, and Per Cent of Variation from Q, by the Hodgkinson and Gordon Formulæ, taken from William E. Merrill's "Iron Truss Bridges for Railroads."

 $h^2 = 16r^2$ , E = 12,215,000, C = 109,801, f = 80,000.

	l + h	k	Q	Variati	on from Q, per cer	ıt, b <del>y</del>
No.	Ratio of Length to Diameter,	Diameter.	Breaking- Weight, by Experiment.	Hodgkinson's Formulæ, Equations (392), (394).	Gordon's Formula, Equation (400), $Q = \frac{80000}{1 + \frac{l^2}{400k^2}}$	Equation (385).
		ins.	lbs. per sq. in.			
117	4	0.520	107674	_ 6	<b>-29.000</b>	<b>— 3.641</b>
118	8	0.500	88964	<b>–</b> 3	-21,000	+ 0.085
119	10	0.777	67502	+13	- 4.000	+19.227
120	13	<b>0.7</b> 68 <sub>.</sub>	55959	+13	- 0.00I	+21.444
121	15	0.500	57321	+ 2	-11,000	+ 5.267
122	15	0.785	50182	+11	+ <del>6000</del>	+20.243
123	15	1.000	51248	+ 9	+ 7.000	+17.741
124	20	0.500	454 <sup>8</sup> 5	- I	-13.000	<b> 1.758</b>
125	20	0.775	45596	- I		<b>- 1.998</b>
126	20	1.022	38770	+12	+ 5.000	+15.257
127	24	0.500	36644	+ 2	<b>—11.000</b>	<b>— 3.291</b>
128	26	0.777	3 <b>28</b> 60	+ +	<b>— 9.000</b>	<b>- 3.503</b>
129	30	0.510	33111	-10	25.000	-22.497
130	30	1.010	25350	+ 5	<b> 3.000</b>	+ 1.231
131	39	0.770	18921	<b>–</b> 8	13.000	—II.284
132	39	1.560	15153	+ 6	+11.000	+10.777
133	40	0.510	18749	- 2	-13.000	-14.241
134	47	1.290	12291	<b>–</b> 3	+ 6	— 1.261
135	61	0.500	8464	+ 6	<b></b> 7.000	<b>—10.881</b>
136	61	0.997	7990	+ 🔒	2.000	<b>-</b> 5.581
137	79	0.770	5274	+ 2	8.000	<b>—12.286</b>
138	119	0.510	2384	+19	<b> 7.000</b>	-12.416
138	119	0.510	2354	+19	<b>—</b> 7.000	-12.410

Equation (392), as used in Table IX., is

$$P = 98922 \frac{k^{3.55}}{\left(\frac{1}{12}l\right)^{1.7}}.$$

See "Iron Truss Bridges for Railroads," p. 26.

For E = 12,215,000, see Stoney's "Theory of Strains," p. 180.

TABLE X.

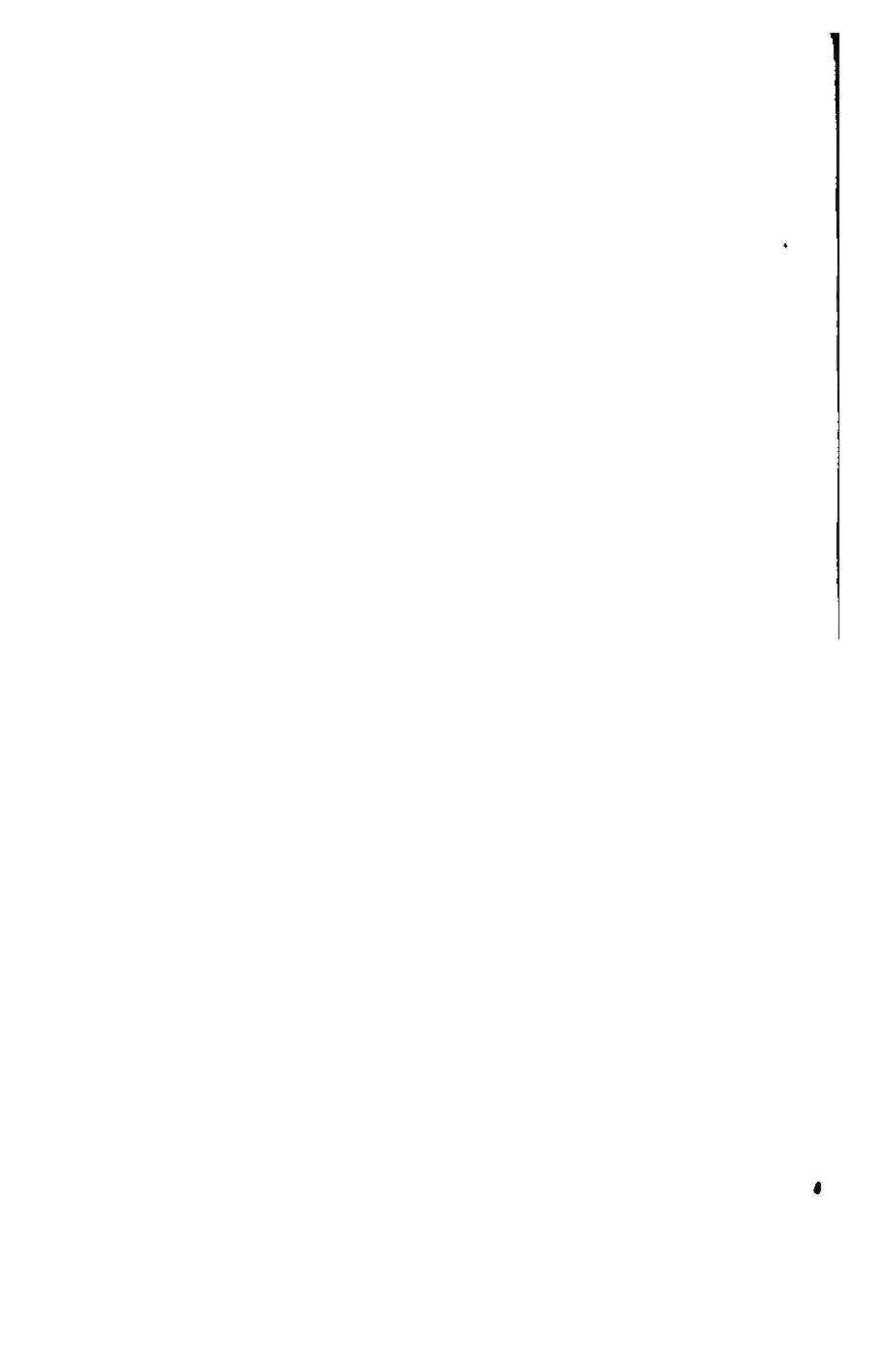
# SOLID CYLINDRICAL PILLARS OF CAST-IRON.

Ends rounded. Hodgkinson's Experiments.

Data, and Per Cent of Variation from Q, by the Hodgkinson and Gordon Formulæ, taken from William E. Merrill's "Iron Truss Bridges for Railroads."

 $k^2 = 16r^2$ , E = 15,268,750, C = 109,801, f = 80,000.

	l÷ k	h	Q	Varia	tion from Q, per c	ent, by
No.	Ratio of Length to Diameter.	Diameter.	Breaking- Weight, by Experiment.	Hodgkinson's Formulæ, Equation (392), $P=33379\frac{\lambda^{3.76}}{\left(\frac{1}{12}l\right)^{1.7}}$	Gordon's Formula, $Q = \frac{80000}{1 + \frac{l^2}{100k^2}}$	Equation, $Q = \frac{C}{1 + \frac{Cl^2}{\pi^2 E r^2}}$
		ins.	lbs. per sq. in.			
139	, 8	0.500	76939	<b>25</b>	-34	<b>—18.269</b>
140	10	0.770	49280	· <b>– 7</b>	<b>—18</b>	+ 2.877
341	13	0.760	38590	<b>—15</b>	25	4.203
142	15	0.497	27124	+ 1	<b>:</b> 1	+11.735
143	15	0.990	25660	+10	6	+18.110
144	20	0.760	20331	-13	<b>—2</b> I	<b>– 4</b> .633
145	20	7.010	19642	<b>- 9</b>	-19	<b>— 1.288</b>
146	20	1.520	17928	+ 3	-10	+ 8.149
147	23	1.290	13187	+ 5	<b>– 7</b>	+16.175
148	26	0.767	14289	23	<b>–29</b>	<b>—13.472</b>
149	30	0.500	9697	-13	-19	<b>– 1.464</b>
150	30	0.990	7931	+ 9	<b>— 2</b>	+20.477
151	31	1.940	7717	+13	<b>– 3</b>	+16.599
152	31	1.960	8051	+14	<b>–</b> 3	+11.763
153	34	1.765	6360	+ 5	<b>—10</b>	+19.262
I 54	34	1.780	7058	+ 6	— <b>10</b>	+ 7.467
155	39	0.7 <b>7</b> 0	5854	<b>– 5</b>	-17	+ 0.137
156	39	1.535	5755	+ 🛊	<b>—16</b>	+ 1.859
257	40	1.520	5985	- <del> </del>	-14	<b>—</b> 6.650
158	47	1.290	4367	- <del> </del>	<b>—18</b>	<b>–</b> 6.000
159	47	1.295	4149	- <del>1</del>	—1 <b>8</b>	<b>— 1.060</b>
160	61	0.500	2745	<b>-</b> 5	<b>—23</b>	<b>-</b> 9.872
161	61	0.990	2471	+ 8	<b>—16</b>	+ 0.121
162	79	0.770	1675	+ 2	-24	-11.105
163	121	0.500	728	+10	<del>-2</del> 5	-12.083



### TABLE XII.

# SOLID SQUARE PILLARS OF PINE.

Data from Bindon B. Stoney's "Theory of Strains in Girders and Similar Structures."

 $k^2 = 12T^2$ . Take E = 1460000, C = f = 5000.

	$l \div k$		$Q = \mathbf{P}$	reaking-Weight	, in Lbs., per S	quare Inc	h.	
No.	Ratio of Length to Least Diam- eter.	Rondelet's Proportionals. Flat Ends.	Brereton's Tests. Ends in Ordinary Manner.	Gordon Formula, $Q = \frac{5000}{1 + \frac{l^2}{250k^2}}$	Hodgkinson's Fermula, $Q = \frac{500Cl^2}{k^2}$	(381). Eduation	Equation (391). One End fully fixed.	Both
184	1	5000	-	_	5000	_	_	-
185	12	4167	_	3176	5000	3126	3959	4349
<b>186</b>	24	2500	-	1513	4940	1471	2437	3126
187	36	1667	-	809	1929	782	1485	2135
<b>188</b>	48	833	_	489	zo85	462	960	1471
189	60	417	-	325	693	313	660	1076
190	72	209	-	230	483	221	478	642
191	10	-	1867	357 <b>°</b>	5000	·3530	4228	4529
192	20	_	1789	1923	5000	1876	2889	3530
193	30	· -	1400	1087	2777	1053	1891	2581
194	40 .	-	1244	676	1563	653	1273	1875

### CHAPTER VIII.

YONS AND WEIGHTS OF ALL THE MEMBERS OF A BRIDGE EXCEPTING THE GIRDERS PROPER.

### The Floor.

! = length of floor, in feet.

q = breadth of floor, in feet.

t = thickness of floor, in feet.

u = weight of one cubic foot of the material, in pounds.

Volume of floor = lqt cubic feet

= 0.012lqt thousand feet, board measure.

F = weight of floor = ulqt pounds. (408)

### The Joists, Longitudinal.

- = length of joist in each panel, in feet.
- = depth of joist, in inches.
- = thickness of joist, in inches.
- = number of equal panels.
- = distance between centres of joists, in feet.
- = number of joists in any panel, each of the two outside ones having the thickness ½b, and being counted as one-half a joist.
- = whole number of joists in the n panels.
- = panel weight of uniform load, in tons.
- = weight of one cubic foot of the material, in pounds.

Weight upon the joists of one panel  $=\frac{ulqt}{n} + 2000L$  pounds,

Uniformly distributed load on one joist =  $\frac{ultg}{n} + \frac{2000gL}{q}$  pounds.

Add weight of joist itself  $=\frac{bdlu_1}{144n}$  pounds.

Total uniform load for each joist is, therefore,

$$\frac{ultg}{n} + \frac{2000gL}{q} + \frac{\delta dlu_x}{144n} = \frac{l}{n}w,$$

where w is the number of pounds per linear foot to be supported by one joist.

Now, by equation (52), we have for the external forces, greatest moment at centre,

$$M = \frac{1}{8}w\left(\frac{l}{n}\right)^2 = \frac{lw}{8} \times \frac{l}{n} = \frac{ul^2tg}{8n^2} + \frac{250glL}{nq} + \frac{bdl^2u_1}{1152n^2} \text{ foot-pounds};$$

and for the internal forces of a rectangular beam, equation (160), the moment of resistance is

$$R = \frac{1}{6}Bbd^2$$
 inch-pounds =  $\frac{1}{12}Bbd^2$  foot-pounds.

Introducing f, the factor of safety, and equating M and  $R \div f$ , we find

$$\frac{Bbd^2}{72f} = \frac{ul^2tg}{8n^2} + \frac{250glL}{nq} + \frac{bdl^2u_1}{1152n^2}.$$
 (409)

Taking the value of B from Table II., and assigning a value to b or d, we may find, from (409), the required depth or thickness of each joist.

If we neglect the weight of the joist itself, which omission the factor of safety may well warrant, the last term in (409) vanishes, and we have at once

Thickness of joist 
$$= b = \frac{9fgl}{n^2qd^2B}(uqlt + 2000nL)$$
.

Depth of joist  $= d = \left\{ \frac{9fgl}{n^2qbB}(uqlt + 2000nL) \right\}^{\frac{1}{2}}$ .

 $J = \text{weight of } (nq + g) \text{ joists } = \frac{bdlqu_t}{144g} \text{ pounds.}$  (410)

In a similar manner may the dimensions and weight of any other joist or beam or stringer be found; that is, by equating the greatest moment due the external forces acting on the beam, to the greatest allowable moment due the internal forces resisting.

124. The Wrought-Iron I Floor Beams, Transverse, supporting the Joists, Floor, and Load.

Let  $d_2$  = depth of beam, in inches.

 $d_{\rm r} =$  depth of web, in inches.

 $d_2 - d_1 =$  depth of two flanges, in inches.

 $b_2$  = breadth of one flange.

 $b_2 - b_1 =$  thickness of web.

 $q_1$  = entire length of beam, in feet.

S = cross-section of beam, in square inches.

n-1 = number of beams in bridge.

m = weight of one cubic inch of wrought-iron, in pounds.

$$D = \frac{F + J + 2000nL}{n} = \text{uniform load on any beam,}$$
 in pounds.

Then, by equation (52),

Moment of external forces =  $M = \frac{3}{2}Dq_1$  inch-pounds.

And, from equation (161),

Moment of internal forces =  $R = \frac{B(b_2d_2^3 - b_1d_1^3)}{6d_2}$  inch-pounds.

Whence, introducing f as the factor of safety,

$$\frac{{}_{3}^{8}Dq_{1}}{{}_{3}^{2}} = \frac{B(b_{2}d_{2}^{3} - b_{1}d_{1}^{3})}{6d_{2}f},$$

$$\frac{b_{2}d_{2}^{3} - b_{1}d_{1}^{3}}{d_{2}} = \frac{9Dq_{1}f}{B} \tag{411}$$

Let us take now the dimensions of the cross-section of a well-proportioned I-beam, as, for instance,  $d_2 = 15$ ,  $d_1 = 12\frac{3}{4}$ ,  $b_2 = 5\frac{3}{8}$ ,  $b_1 = 4\frac{3}{4}$ , and express the relation

$$d_2 = \frac{10}{17}d_1 = \frac{120}{48}b_2 = \frac{60}{19}b_1,$$

$$\therefore d_1 = \frac{17}{10}d_2, b_2 = \frac{43}{120}d_2, b_1 = \frac{19}{68}d_2.$$

Therefore (411) becomes

$$\begin{bmatrix} \frac{43}{120} - \frac{19}{60} (\frac{17}{20})^3 \end{bmatrix} d_2^3 = \frac{9Dq_1 f}{B},$$

$$\therefore d_2 = 3.80122 \left( \frac{Dq_1 f}{B} \right)^{\frac{1}{2}}.$$
(412)

Area of section = 
$$S = b_2 d_2 - b_3 d_4 = \frac{107}{1200} d_2^2$$
  
= 1.28839 $\left(\frac{Dq_1 f}{B}\right)^{\frac{3}{2}}$ . (413)

$$P = \text{weight of floor beams} = 12q_1 m(n-1)S$$

$$= 15.46068 mq_1(n-1) \left(\frac{Dq_1 f}{B}\right)^{\frac{1}{2}}.$$
(414)

If the beam actually used has a form of cross-section varying materially from that here assumed, the co-efficient of (412) must be made to conform thereto.

We may compensate for the omission of the beam's own weight from the formula, first, by selecting from the manufacturer's list of beams that one whose depth agrees most nearly with our computed depth *above* it; and second, by using, in the calculation, the entire length of beam, instead of the net length between bearings.

Having thus employed the formula to determine the depth of beam required for the given load, the weight may be taken from the manufacturer's tables. Indeed, the manufacturer's tables of strength may be used without this calculation, whenever they are known to be trustworthy, by selecting the depth of beam corresponding to the required length and "safe load."

125. The System of Lateral Support. — This system includes whatever arrangement of struts, ties, and braces is employed to prevent a lateral bending of the girders, and their rotation about their points of support.

The arrangement must manifestly vary with the form and height of girder; a high girder with straight chords allowing a complete horizontal trussing overhead and under the floor, while arched top chords allow only a partial head-bracing, and low girders for "through" bridges can only be laterally braced from below.

In all cases, the horizontal systems, top and bottom, should be rigidly connected with the girders, whether angle braces are employed or not. For high girders with straight chords, there are generally used, a strut at every pair of opposite top joints, n + 1 in number, and a pair of diagonal ties at the top and bottom of each panel, 4n in number.

The proportions of these members may be computed in the same manner as the proportions are found for a girder uniformly loaded, using the assumed pressure of wind against the side of the bridge and load as the uniform horizontally (or otherwise) acting load.

For girders admitting full head-bracing, we thus compute:

q =length of horizontal strut, in feet.

$$\sqrt{q^2 + \left(\frac{l}{n}\right)^2}$$
 = length of horizontal diagonal, in feet.

 $S_r = \text{cross-section of each strut, in square inches,}$  Assumed

 $S_2$  = cross-section of each diagonal, in square or inches, computed.

m = weight of one cubic inch of wrought-iron.

$$U = \text{weight of horizontal struts} = 12qm(n + 1)S_1.$$
 (415)

$$X = \text{weight of horizontal diagonals} = 48mnS_2\sqrt{q^2 + \frac{l^2}{n^2}}$$
. (416)

126. Finally, there should be added whatever weight of wood or iron is not included in the foregoing specifications, but is employed in the actual completion and equipment of the structure. Call this weight p pounds to the panel; then we have

$$Y = \text{weight of residue} = np \text{ pounds.}$$
 (417)

127. Take K = weight of bridge exclusive of the girders, in pounds; then

$$K = F + J + P + U + X + Y$$
 pounds. (418)

And if G = weight of girders, in pounds,

Weight of bridge = 
$$2000nW = K + G$$
 pounds. (419)

# CHAPTER IX.

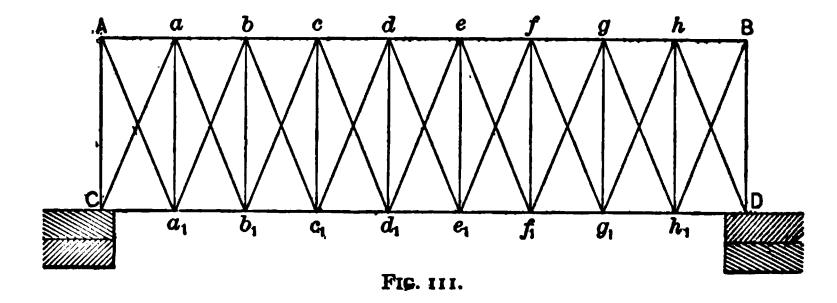
OPEN GIRDERS WITH EQUAL AND PARALLEL STRAIGHT CHORDS.

CLASS IX.

# SECTION 1.

The Pratt Truss of Single System and Uniform Live Load. — Wind Pressure.

128. Strains in Terms of the Structure's Unknown Weight. — Let Fig. 111 represent a girder, or built beam, having a discontinuous or open web, and its flanges or chords



in straight horizontal lines, AB, CD. Let the vertical members, or posts, be in compression, and the inclined members, or diagonals, be in tension. We then have a girder which has been called the Murphy, Whipple, or Pratt truss of single intersection, since the diagonals traverse but a single panel or division of the girder.

Take n =number of panels.

l = length of girder, in feet.

 $l \div n = c = length of one panel.$ 

h = height of girder between centres of chords, in feet.

 $\phi$  = inclination of diagonals to chords.

$$\therefore \tan \phi = \pm \frac{h}{c} = \pm \frac{nh}{l}, \quad \sin \phi = \frac{h}{\sqrt{c^2 + h^2}}, \quad \cos \phi = \frac{c}{\sqrt{c^2 + h^2}}.$$

$$\sqrt{c^2 + h^2} = \frac{h}{\sin \phi} = \frac{c}{\cos \phi} = \frac{l}{n \cos \phi} = \text{length of a diagonal.}$$

$$\sqrt{c^2 + h^2} = \sqrt{\frac{l^2}{n^2} + h^2} = \frac{1}{n}\sqrt{l^2 + n^2h^2} = \text{length of a diagonal.}$$

Assume the entire weight of the structure supported by the girder, including the girder's own weight, to be uniform throughout, and equal to nW tons applied at the lower joints; viz.,  $\frac{1}{2}W$  at C,  $\frac{1}{2}W$  at D, and W at each of the other n-1 joints, apices, or panel points,  $a_1$ ,  $b_1$ ,  $c_2$ , etc. W is called the panel weight, or apex load, due to the permanent weight of the structure.

Total pressure at C or  $D = \frac{1}{2}nW = \text{resistance}$  of pier to the permanent load.

Assume also a uniform moving-load, nL, advancing by apex loads, L, from left to right, upon and over the girder. We then have total weight = n(W + L) tons; weight at each apex = W + L tons when fully loaded.

With these data, we proceed to find the greatest strains developed in each member of the girder by the permanent load nW, and the uniform moving-load nL.

(a) To find the moment at each joint due the entire weight n(W + L), and thence the horizontal strain in chords by equation (95).

 $\cdot H = M + h =$ moment divided by height.

Equation (65) applies here if for W we put W + L, and we have

$$M_{a} = \frac{(W+L)l}{2n}(n-1) \times 1,$$

$$\therefore H_{a} = \frac{(W+L)l}{2nh}(n-1) \times 1 = \text{strain on } Aa, a_{1}b_{1};$$

$$M_{b} = \frac{(W+L)l}{2n}(n-2) \times 2,$$

$$\therefore H_{b} = \frac{(W+L)l}{2nh}(n-2) \times 2 = \text{strain on } ab, b_{1}c_{1};$$

$$M_c = \frac{(W+L)l}{2n}(n-3) \times 3,$$

$$\therefore H_c = \frac{(W+L)l}{2nh}(n-3) \times 3 = \text{strain on } bc, c_1d_1;$$

$$M_d = \frac{(W+L)l}{2n}(n-4) \times 4,$$

$$\therefore H_d = \frac{(W+L)l}{2nh}(n-4) \times 4 = \text{strain on } cd, d_1e_1;$$

$$M_{h} = \frac{(W+L)l}{2n}[n-(n-1)](n-1),$$

$$\therefore H_{h} = \frac{(W+L)l}{2nh}[n-(n-1)](n-1) = \text{strain on } hB, g_{1}h_{1};$$

where H is the greatest horizontal strain, in tons, at the successive joints; the strain on each chord being assumed to act at the centre or axis of the chord, whose depth is small compared with h.

(b) To find the compression on verticals, and the tension on diagonals, due to permanent load, nW, alone.

From equation (65), dividing by h, and from the formulæ for Class IX., article 49,

$$H_{A} = 0;$$

$$H_{a} = \frac{W7}{2nh}(n-1) \times 1,$$

$$\therefore \Delta H = H_{a} - H_{A} = \frac{W7}{2nh}(n-1) = \text{hor. component of } Aa_{1};$$

$$H_{b} = \frac{W7}{2nh}(n-2) \times 2,$$

$$\therefore H_{b} - H_{a} = \frac{W7}{2nh}(n-3) = \text{hor. component of } ab_{1};$$

$$H_{c} = \frac{W7}{2nh}(n-3) \times 3,$$

$$\therefore H_{c} - H_{b} = \frac{W7}{2nh}(n-5) = \text{hor. component of } bc_{1};$$

$$\vdots H_{b} - H_{b} = \frac{W7}{2nh}[n-(n-1)](n-1),$$

$$\therefore H_{b} - H_{b} = \frac{W7}{2nh}[-(n-1)] = \text{hor. component of } h_{1}B.$$

Therefore, from the triangle of forces, equations (3), the vertical components are

$$Z = \Delta H \tan \phi = \pm \Delta H \frac{nh}{l}; \qquad (420)$$

$$\therefore Z_A = \frac{1}{2}W(n-1) = \text{compression on } AC \text{ or } BD,$$

$$Z_a = \frac{1}{2}W(n-3) = \text{compression on } aa_1 \text{ or } hh_1,$$

$$Z_b = \frac{1}{2}W(n-5) = \text{compression on } bb_1 \text{ or } gg_1,$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$Z_B = \frac{1}{2}W(n-1) = \text{compression on } BD \text{ or } AC.$$

And the strain Y along any diagonal is

$$\Delta H + \cos \phi$$
.

or

$$Z + \sin \phi = \frac{Z\sqrt{l^2 + n^2h^2}}{nh}; \qquad (421)$$

$$Y_A = \frac{W}{2 \sin \phi} (n - 1) = \text{tension on } Aa_1 \text{ or } Bh_1,$$

$$Y_a = \frac{W}{2 \sin \phi} (n - 3) = \text{tension on } ab_1 \text{ or } hg_1,$$

$$Y_b = \frac{W}{2 \sin \phi} (n - 5) = \text{tension on } bc_1 \text{ or } gf_1,$$

$$Y_B = \frac{W}{2 \sin \phi} \left[ -(n - 1) \right] = \text{tension on } Bh_1 \text{ or } Aa_1.$$

(c) Maximum strain on verticals and diagonals from moving-load, nL, alone.

To find this strain  $Z_L$ , we subtract equation (64) from (68), divide remainder by h for greatest difference of horizontal strains at adjacent joints, and multiply the quotient by  $\tan \phi$ ; thus, after putting L for W, the difference between (68) and (64) is

$$\frac{Ll}{2n^2} \times r(r+1) = \text{maximum } \Delta Hh \text{ (say)},$$

$$\therefore Z_L = \frac{\tan \phi}{h} \times \frac{Ll}{2n^2} \times r(r+1) = \frac{L}{n} \times \frac{r(r+1)}{2}, \quad (422)$$

where r is the number of apex loads on the girder as the moving-load advances, and  $Z_L$  is the compression on the (r + 1)<sup>th</sup> vertical;

.. 
$$Z_b = \frac{L}{n} \times 1 = \text{compression on } bb_1, \qquad r = 1;$$

$$Z_c = \frac{L}{n} \times 3 = \text{compression on } cc_1, \qquad r = 2;$$

$$Z_d = \frac{L}{n} \times 6 = \text{compression on } dd_1, \qquad r = 3;$$

$$Z_e = \frac{L}{n} \times 10 = \text{compression on } ee_1, \qquad r = 4;$$

$$Z_B = \frac{L}{n} \times \frac{(n-1)n}{2} = \text{compression on } BD, \qquad r = n-1.$$

The greatest strain on diagonals due to moving-load, nL, is

$$Y = Z_L + \sin \phi; \qquad (423)$$

$$Y_b = \frac{L}{n \sin \phi} \times 1 = \text{tension on } a_1 b, \qquad r = 1;$$

$$Y_c = \frac{L}{n \sin \phi} \times 3 = \text{tension on } b_1 c, \qquad r = 2;$$

$$Y_d = \frac{L}{n \sin \phi} \times 6 = \text{tension on } c_1 d, \qquad r = 3;$$

$$Y_e = \frac{L}{n \sin \phi} \times 10 = \text{tension on } d_1 e, \qquad r = 4;$$

$$Y_B = \frac{L}{n \sin \phi} \times \frac{(n-1)n}{2} = \text{tension on } h_1 B, \quad r = n-1.$$

(d) Combining the strains due nW and nL, and, for convenience, writing N for  $\frac{(W+L)l}{2nh}$ , we find, for any number of panels:—

# (1-u) M. 2 (n-2) N. 3 (n-5) N. 4 (n-4) N. (n-u) N. (n-u) M. (n-u)

# MAXIMA STRAINS IN PRATT TRUSS.

FIG. 112.

2 (n-2) N.

3 (n-3) N.

### Uniform Dead and Live Loads.

Loads applied at lower joints:—

(u-1) N.

W =panel weight of dead load.

L =panel weight of live load.

l = length of truss from centre to centre of end pins.

h =height of truss from centre to centre of pins.

n = number of panels.

0

$$\sin \phi = \frac{nh}{\sqrt{l^2 + n^2h^2}}, \quad N = \frac{(W + L)l}{2nh} = \frac{W + L}{2\tan \phi}.$$

129. Weight of the Structure determined. — (a) To find the weight of the top chord.

Suppose  $Q \div f$  to be the greatest allowable pressure to the square inch of section of top chord, and Q to be of the same denomination with W and L; and suppose f to be a number called the factor of safety. Q is known as the breaking-weight

J

of a column of the given material, having the length of one panel, and the cross-section of the top chord for any given panel.

Let m = weight of one cubic inch of the material, in pounds.

We then have, for each one of the equal panel lengths of the top chord,

Area of section 
$$= \frac{fH}{Q} \text{ square inches,}$$
Volume of one panel length 
$$= \frac{12flH}{Qn} \text{ cubic inches,}$$
Weight of one panel length 
$$= \frac{12mfl}{Qn} H \text{ pounds,}$$

Weight of top chord 
$$=\frac{12mfl}{Qn}\Sigma H$$
 pounds.

From (a) of the preceding article we find

$$\Sigma H = \frac{(W+L)l}{2nh} \begin{cases} n[1+2+3+4+\dots(n-1) \text{ terms}] \\ -[1^2+2^2+3^2+4^2+\dots(n-1) \text{ terms}] \\ +\frac{1}{4}n^2 \text{ for } n \text{ even, } +\frac{1}{4}(n^2-1) \text{ for } n \text{ odd.} \end{cases}$$

$$= \frac{(W+L)l}{2nh} \times \frac{n}{12}(2n^2+3n-2), n \text{ even;}$$

$$= \frac{(W+L)l}{2nh} \times \frac{1}{12}(2n^3+3n^2-2n-3), n \text{ odd.}$$

Substituting these values for  $\Sigma H$ , we have

Weight of top chord 
$$= \frac{mf^{2}(W+L)}{2Qnh}(2n^{2}+3n-2)$$

$$(n \text{ even}),$$

$$= \frac{mf^{2}(W+L)}{2Qn^{2}h}(2n^{3}+3n^{2}-2n-3)$$

$$(n \text{ odd}).$$
(424)

(b) Similarly, if  $T \div f =$  the greatest allowable tensile strain, we find

Weight of bottom chord =  $\frac{12mfl}{Tn}\Sigma H$  pounds.

$$\Sigma H = \frac{(W+L)!}{2nh} \begin{cases} +2(n-1) \text{ for two end panels,} \\ n[1+2+3+4+\dots(n-1) \text{ terms}] \\ -[1^2+2^2+3^2+\dots(n-1) \text{ terms}] \\ -\frac{1}{4}n^2 \text{ for } n \text{ even, } -\frac{1}{4}(n^2-1) \text{ for } n \text{ odd.} \end{cases}$$

$$= \frac{(W+L)l}{2nh} \times \frac{1}{12}(2n^3 - 3n^2 + 22n - 24) \text{ when } n \text{ is even,}$$

$$= \frac{(W+L)l}{2nh} \times \frac{1}{12}(2n^3 - 3n^2 + 22n - 21) \text{ when } n \text{ is odd.}$$

Weight of bottom chord
$$= \frac{mfl^{2}(W + L)}{2Tn^{2}h}(2n^{3} - 3n^{2} + 22n - 24)$$

$$(n \text{ even}),$$

$$= \frac{mfl^{2}(W + L)}{2Tn^{2}h}(2n^{3} - 3n^{2} + 22n - 21)$$

$$(n \text{ odd}).$$

(c) In finding the weight of the verticals, let  $Q_i \div f$  be the allowed working unit strain in compression; then

Area of section 
$$= \frac{f(Z_W + Z_L)}{Q_I} \text{ square inches,}$$
Volume of one strut 
$$= \frac{12fh(Z_W + Z_L)}{Q_I} \text{ cubic inches,}$$
Weight of one strut 
$$= \frac{12mfh(Z_W + Z_L)}{Q_I} \text{ pounds,}$$
Weight of all verticals 
$$= \frac{12mfh(\Sigma Z_W + \Sigma Z_L)}{Q_I} \text{ pounds.}$$

Hence, from the strain sheet, Fig. 112, using the proper limits of summation, we derive, when n is even,

$$\Sigma Z_{W} = 2 \times \frac{W}{2} \{ n(\frac{1}{2}n) - [1 + 3 + 5 + 7 + \dots (\frac{1}{2}n) \text{ terms}] \}$$

$$= \frac{Wn^{2}}{4},$$

$$\Sigma Z_{L} = 2 \times \frac{L}{2n} \{ n^{2}(\frac{1}{2}n) - n[1 + 3 + 5 + 7 \dots (\frac{1}{2}n) \text{ terms}] + 2[1 + 3 + 6 + 10 + \dots (\frac{n}{2} - 1) \text{ terms}] + \frac{1}{2} (n - \frac{n}{2}) [n - (\frac{n}{2} + 1)] \text{ for middle strut} \}$$

$$= \frac{L}{n} \times \frac{1}{24} (7n^{3} + 3n^{2} - 10n).$$

But when n is odd, we thus sum,

$$\Sigma Z_{W} = 2 \times \frac{W}{2} \left\{ n \frac{n-1}{2} - \left( 1 + 3 + 5 + 7 + \dots \frac{n-1}{2} \text{ terms} \right) \right\}$$

$$= \frac{W(n^{2} - 1)}{4}.$$

$$\Sigma Z_{L} = 2 \times \frac{L}{2n} \left\{ n^{2} \frac{n-1}{2} - n \left( 1 + 3 + 5 + 7 + \dots \frac{n-1}{2} \text{ terms} \right) + 2 \left( 1 + 3 + 6 + 10 + \dots \frac{n-3}{2} \text{ terms} \right) \right\}$$

$$= \frac{L}{n} \times \frac{1}{24} (7n^{3} - 3n^{2} - 7n + 3).$$

Wherefore

Weight of verticals = 
$$\frac{3mfh Wn^{2}}{Q_{1}} + \frac{mfh L}{2 Q_{1}} (7n^{2} + 3n - 10)$$

$$(n \text{ even}),$$

$$= \frac{3mfh W(n^{2} - 1)}{Q_{1}}$$

$$+ \frac{mfh L}{2 Q_{1}n} (7n^{3} - 3n^{2} - 7n + 3)$$

$$(n \text{ odd}).$$

$$(n \text{ odd}).$$

(d) In determining the weight of the diagonals in terms of the unknown weight of the structure, nW, we shall disregard the effect of the permanent weight, nW, upon the strains developed in the *counter* diagonals by the moving-load, nL.

By so doing, the value of W comes out a little greater than strict theory requires; but in general practice the "counters" are inserted somewhat in excess of theoretical demands.

When, however, W shall have been thus determined, the strains upon all the members are to be computed according to the strain sheet, Fig. 112.

Strain on any diagonal due to L is  $Y_L$ .

Area of cross-section 
$$=\frac{fY_L}{T}$$
 square inches,

Volume of one diagonal = 
$$\frac{12fh}{T\sin\phi}Y_L$$
 cubic inches,

Weight of one diagonal 
$$=\frac{12mfh}{T\sin\phi}Y_L$$
 pounds.

From Fig. 112,

$$\Sigma Y_L = 2 \times \frac{L}{n \sin \phi} [1 + 3 + 6 + 10 + \dots (n - 1) \text{ terms}]$$

$$= \frac{2L}{n \sin \phi} \times \frac{n(n^2 - 1)}{6},$$

therefore weight of diagonals due uniform moving-load, \*L, alone is

$$\frac{4mfhL}{T\sin^2\phi}(n^2-1). \tag{427}$$

The weight of the diagonals due to the dead load, nW, is manifestly to be derived from the weight of the verticals due dead load if for h we put  $h \div \sin^2 \phi$ , and for  $Q_1$  we put T.

$$\therefore \text{ Weight of diagonals} = \frac{4mfhL(n^2 - 1)}{T\sin^2\phi} + \frac{3mfhWn^2}{T\sin^2\phi}$$

$$= \frac{4mfhL(n^2 - 1)}{T\sin^2\phi} + \frac{3mfhW(n^2 - 1)}{T\sin^2\phi}$$

$$= \frac{4mfhL(n^2 - 1)}{T\sin^2\phi} + \frac{3mfhW(n^2 - 1)}{T\sin^2\phi}$$

$$= \frac{4mfhL(n^2 - 1)}{T\sin^2\phi} + \frac{3mfhW(n^2 - 1)}{T\sin^2\phi}$$

(e) Taking the sum of the weights thus found, we have, when n is even, total weight of girder, in pounds,

$$G = \frac{mfl^{2}(W+L)}{2nh} \left( \frac{2n^{2}+3n-2}{Q} + \frac{2n^{3}-3n^{2}+22n-24}{Tn} \right) + 3mfhn^{2}W \left( \frac{1}{Q_{i}} + \frac{1}{T\sin^{2}\phi} \right) + mfhL \left\{ \frac{7n^{2}+3n-10}{2Q_{1}} + \frac{4(n^{2}-1)}{T\sin^{2}\phi} \right\}.$$
 (429)

But when n is odd, total weight of girder, in pounds, is

$$G = \frac{mfl^{2}(W+L)}{2n^{2}h} \left( \frac{2n^{3}+3n^{2}-2n-3}{Q} + \frac{2n^{3}-3n^{2}+22n-21}{T} \right) + 3mfh(n^{2}-1)W\left( \frac{1}{Q_{1}} + \frac{1}{T\sin^{2}\phi} \right) + mfhL \left\{ \frac{7n^{3}-3n^{2}-7n+3}{2Q_{1}n} + \frac{4(n^{2}-1)}{T\sin^{2}\phi} \right\}.$$
(430)

EXAMPLE I. — Wrought-iron girder of 6 equal panels. Take n = 6, l = 60 feet, h = 10 feet, f = 4,  $m = \frac{5}{18}$  pound, L = 8 tons, T = 24 tons, Q = 16 tons,  $Q_1 = 12$  tons,  $\tan \phi = 1$ ,  $\sin \phi = \frac{1}{2}\sqrt{2} = 0.70711$ . Therefore, from (429),

$$G = 483.333W + 4267$$
 pounds,

equal to 2000nW if nW is the girder's own weight in tons.

.. Panel weight of girder = W = 0.3704775 ton, Total weight of girder = nW = 2.2228650 tons. 130. But if the structure is a bridge having two equal girders whose combined weight is G, and an additional permanent weight of K pounds, then the weight of the bridge is

$$2000nW = K + G$$
 pounds,

as shown by equation (419).

Continuing the first example of article 129, we compute K as follows:—

For the floor, we have l = 60 feet = length.

Take q = 18 feet = breadth.

$$t = \frac{2.5}{12}$$
 feet = thickness.

u = 54 pounds = weight of one cubic foot of oak.

From (408),

Weight of floor = 54 × 60 × 18 ×  $\frac{2.5}{12}$  = 12150 pounds = F.

The joists:

l + n = 10 feet = panel length of joist.

Take b = 3 inches = thickness.

g = 2 feet = space between centres.

 $q \div g = 9$  = number of joists in each panel.

 $nq \div g = 54 = \text{number of joists in bridge.}$ 

 $u_{\rm r}=54=u$ .

B = 10,600.

f=9

Then, by article 123, we have

$$d = \left\{ \frac{9 \times 9 \times 60 \times 2}{6^2 \times 18 \times 3 \times 10600} \left( 54 \times 18 \times 60 \times \frac{2.5}{12} + 2000 \times 6 \times 8 \right) \right\}^{\frac{1}{4}}$$
  
= 7.1424 inches.

Call d = 8 inches,

$$\therefore \text{ Weight of joists} = \frac{3 \times 8 \times 60 \times 18 \times 54}{144 \times 2} = 4860 \text{ pounds} = J.$$

For the iron I-beams, we have, from article 124,

$$D = \frac{F + J + 2000nL}{n} = 18835$$
 pounds.

Take  $q_1 = 19$  feet = entire length of beam.

f = 4 = factor of safety.

B = 52,567, from Table II.

Whence, by equation (412),

Required depth of beam = 
$$d_2 = 3.80122 \left( \frac{18835 \times 19 \times 4}{52567} \right)^{\frac{1}{2}}$$
  
= 11.435 inches.

Call  $d_2 = 12$  inches; then, by (413),

Area of section = 
$$S = 1.28839 \left( \frac{18835 \times 19 \times 4}{52567} \right)^{\frac{2}{3}} \times \left( \frac{12}{11.435} \right)^{2}$$
  
= 12.84 square inches,

since similar sections are to each other as the squares of their like dimensions.

Now this cross-section, 12.84, agrees very nearly with that of the "12-inch light I-beam" of the Union Iron Mills, Pitts-burgh, Penn., whose weight is 42 pounds to the foot, and area =  $42 \times \frac{8}{10} = 12.6$  square inches.

Using this beam, we then have

Weight of 5 floor beams =  $5 \times 19 \times 42 = 3990$  pounds = P.

Use full head trussing; the struts to be composed of two T-bars, each  $5\frac{1}{2}$  pounds to the foot, latticed with  $1\frac{1}{4} \times \frac{1}{4}$  inch

braces, at 45 degrees, the whole weighing  $12\frac{1}{2}$  pounds to the running foot; length = 18 feet.

Weight of 
$$(n + 1)$$
 horizontal struts =  $7 \times 18 \times 12\frac{1}{2}$   
=  $1575$  pounds =  $U$ .

Let the horizontal diagonal ties be  $1\frac{1}{8}$  inches in diameter, weighing 3.359 pounds to the foot. Then

Weight of 24 horizontal ties = 
$$24 \times 3.359\sqrt{10^2 + 18^2}$$
  
=  $1660$  pounds =  $X$ .

Call the residue 100 pounds to the panel; that is, in all = 600 pounds = Y.

.. 
$$K = F + J + P + U + X + Y = 24835$$
 pounds,  
 $G = \text{weight of girders} = 4267 + 483.333 W$ ,  
 $K + G = \text{weight of bridge} = 29102 + 483.333 W$   
= 12000 W pounds;

.. Panel weight of bridge = W = 2.526947 tons, Total weight of bridge = nW = 15.161682 tons.

Panel weight of dead load on each girder = 1.26347 tons, Panel weight of live load on each girder = 4 tons.

 $\frac{1}{2}(W+L) = 5.26347$  tons = total panel weight for one girder.

Putting this value, 5.26347 tons, for W + L, in the expression for N, article 128, (d), we find

$$N = \frac{5.26347l}{2nh} = \frac{5.26347 \times 60}{2 \times 6 \times 10} = 2.63174 \text{ tons.}$$

And from the strain sheet, Fig. 112, the greatest chord strains are

$$H_1 = 2.63174 \times 5 \times 1 = 13.15870$$
tons,  
 $H_2 = 2.63174 \times 4 \times 2 = 21.05392$ tons,  
 $H_3 = 2.63174 \times 3 \times 3 = 23.68566$ tons.

Putting  $\frac{1}{2}W = 1.26347$  for W, and  $\frac{1}{2}L = 4$  for L, the same strain sheet gives, for each of two girders:—

Greatest compression on verticals:

$$Z_1 = 0.63174 \times 5 + 0.33333 \times 5 \times 6 = 13.15870 \text{ tons,}$$
  
 $Z_2 = 0.63174 \times 3 + 0.33333 \times 4 \times 5 = 8.56188 \text{ tons,}$   
 $Z_3 \doteq 0.63174 \times 1 + 0.33333 \times 3 \times 4 = 4.63174 \text{ tons,}$   
 $Z_4 = 0.33333 \times 2 \times 3 = 2.00000 \text{ tons.}$ 

Also, for the diagonals:

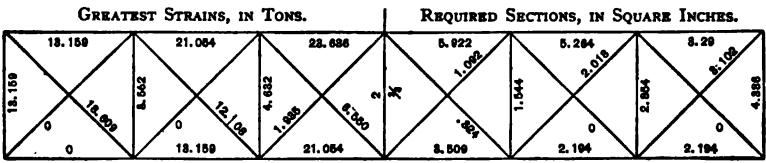
$$\frac{L}{n \sin \phi} \text{ becomes } \frac{4}{6 \sin \phi} = 0.94281 \text{ ton,}$$

$$\frac{W}{2 \sin \phi} \text{ becomes } \frac{0.63174}{\sin \phi} = 0.89342 \text{ ton.}$$

And, from Fig. 112:—

Greatest strain on diagonals:

$$Y_1$$
, counter, = 0 - 0.89342  $\times$  5 < 0 ton;  
 $Y_2$ , counter, = 0.94281  $\times$  1 - 0.89342  $\times$  3 < 0 ton;  
 $Y_3$ , counter, = 0.94281  $\times$  3 - 0.89342  $\times$  1 = 1.93501 tons;  
 $Y_4$ , main, = 0.94281  $\times$  6 + 0.89342  $\times$  1 = 6.55028 tons;  
 $Y_5$ , main, = 0.94281  $\times$  10 + 0.89342  $\times$  3 = 12.10836 tons;  
 $Y_6$ , main, = 0.94281  $\times$  15 + 0.89342  $\times$  5 = 18.60925 tons.



Load applied at Bottom Joints. Diagonals in Tension. — Class IX. Fig. 113.

131. Now, it is very evident that the bridge would depend entirely upon the floor system for stability in a longitudinal direction if we should omit the bottom chords and the counter ties, which are marked as receiving no strain in the *end* panels.

It is therefore usual to insert these members in the end panels, and also, in this style of girder, to stiffen the bottom chords by cross-bracing in the end panels, so that each bottom chord may there act as a strut.

Some builders place counters in all panels, even where the assumed behavior of the given load does not require them. By so doing they provide for *concentrated* loads greater than the assumed uniform apex load, as well as enhance the symmetry of the structure.

In short-span bridges, as in the present example, some of the vertical struts require a greater cross-section than the actual downward pressure upon them would indicate; for, besides this pressure along the axis of the strut, it should be able to resist probable lateral blows from the travel of the road, even though every strut be protected from ordinary collision with hubs.

To provide for the increase of bridge weight from these sources, above the weight computed from the given load nL, we have added the last term, Y, of K, which term should be large enough to cover every thing not otherwise included.

No absolutely definite rule can be given for the size of these parts, but the smallest counter ties should be so large as not to look wiry, say not less than one inch in cross-section; the bottom chord in the first panel may equal in size that of the second panel; and the size of any vertical strut should enable it to resist such lateral shocks as are probable in the situation.

132. Thus far the panel weights, W and L, have been assumed to be applied at the lower joints only. In the nature of the case, however, it is plain that the weight of the top chords and the system of head-bracing, as also the weight of the girder diagonals, can only reach the bottom joints through the vertical struts. But as the weight of these members is small compared with the whole weight of the bridge, and as

the calculation is a little more simple when W is applied at one point instead of two, it is usual to make the above assumption.

It is proper, however, in this place, to indicate the changes to be made in the strain sheet, Fig. 112, by changes in the distribution of the loads.

1st, Suppose the panel weight, L, of live load, and the panel weight of the floor system, and half the panel weight of the girders, to be applied at each lower joint, and the other half of the girders' weight, and the system of head-trussing, to be uniformly distributed at the upper joints.

We have G = weight of girders, in pounds.

 $G \div 2n =$ one-half panel weight of girders, in pounds. Take A = panel weight of head-bracing,

.. Load at each lower joint 
$$= L + W - A - \frac{G}{2n}$$
,

Load at each upper joint  $= A + \frac{G}{2n}$  pounds.

The strain sheet, Fig. 112, applies to this case if to the compression on each vertical we add  $A + \frac{G}{2n}$  pounds, but to each end post  $\frac{1}{2}(A + \frac{G}{2n})$ . And the additional weight of the verticals due to this change of loading is

$$\frac{12mfh}{Q_i}\left(A+\frac{G}{2n}\right)n \text{ pounds},$$

which is to be added to the weight of verticals in equation (426), and, of course, to the second member of (429) and (430), and thence a new expression for G be found.

2d, Suppose we have a "deck" bridge, and that both W and L are applied at the upper joints.

Then to each vertical compression given in strain sheet, Fig. 112, we must add W + L; and the additional weight of the verticals is, with  $\frac{1}{2}(W + L)$  on each end post,

$$\frac{12mfh}{Q_i}(W+L)n \text{ pounds,}$$

to be placed in the second member of (429) and (430), provided the bridge has its points of support at the bottom, as in the figure; but if the points of support are at the ends of the upper chords, then no end posts are required, and their weight may be deducted from the second member of (429) and (430), and

$$\frac{12mfh}{O_1}(W+L)(n-1)$$

be added.

3d, In case of the deck bridge, if we suppose half the weight of the girders, and the weight,  $nA_1$ , of the bottom horizontal bracing, to be applied uniformly at the bottom joints, while the remainder of the loading is applied at the upper joints, we must then add to the pressure on each vertical, Fig. 112,

$$L + W - A_i - \frac{G}{2n}$$
 pounds,

instead of W + L. And the additional weight of girders from this source is

$$\frac{12mfh}{Q_1}\left(L+W-A_1-\frac{G}{2n}\right)n$$
 pounds,

or

$$\frac{12mfh}{Q_1}\left(L+W-A_1-\frac{G}{2n}\right)(n-1)$$
 pounds,

minus the weight of the end posts, according as the girders are supported at bottom or at top.

133. The deck bridge requires, especially when its points of support are at the bottom, a thorough system of lateral sway-bracing, which may be made by inserting diagonals between each top chord and the bottom chord of the opposite girder at the panel joints, in addition to the horizontal systems already provided for.

The proper size of these diagonals can be determined by calculation when the applied external forces are given, so as to conform to the magnitude, situation, and uses of the structure.

Their weight is to be included in the value of K, the constant part of the bridge weight.

134. To determine the best number of panels, n, and the best height, h, of girder, for a bridge of given span, l, and given moving panel load, L, we may find, by means of equations (419), (429), and (430), an expression for W, the panel weight of bridge, in terms of n and h; then, putting  $\left(\frac{dW}{dn}\right) = 0$ , and

 $\left(\frac{dW}{dh}\right) = 0$ , we shall have two simultaneous equations which will yield those values of n and h that will render W a minimum. But in practice it will be found more convenient, since n is always an integer, and the two simultaneous equations are of a high degree, to find W in terms of h alone for several different values of n, presumably including the best, and then to find from  $\frac{dW}{dh} = 0$ , for each value of n, the value of h which

renders W least. It is evident that the values of n and k which simultaneously render W least are the values sought. For the present purpose, we must, of course, retain n and k, or their equivalents, wherever they occur in both K and G. Let us, therefore, re-examine the several terms of K and G, and put them into suitable form for general application.

The value of F, the weight of floor, (408), is independent of n and h, and requires no change.

If for joists we call 
$$d = b^2$$
, (431)

we shall have a good ratio of breadth, b, to depth, d; and, in (410),  $bd = b^3$ , and

$$b = \left\{ \frac{gfgl}{n^2qB} (ulqt + 2000nL) \right\}^{\frac{1}{6}}, \qquad (432)$$

$$\therefore J = \frac{lqu_1}{144g} \left\{ \frac{9fgl}{n^2qB} (ulqt + 2000nL) \right\}^{\frac{3}{8}}, \qquad (433)$$

which is the weight of the joists, in pounds. Restoring the value of D, we write, for (414),

$$P = 15.46068 mq_1(n-1) \left\{ \frac{(F+J+2000nL)q_1f}{nB} \right\}^{\frac{2}{3}} \text{ pounds, (434)}$$

equal to weight of (n - 1) wrought-iron I-beams having the proportions assumed in deriving equation (412).

of wind horizontally against the side of each open girder and its moving-load, or against the entire side of each wholly covered structure, we find the strains due to wind, in the chords and entire lateral system, by making the proper changes in the strain sheet, Fig. 112.

For any through bridge of Class IX., let the uniform wind pressure to be resisted by the top or bottom lateral system be  $W_1 = \frac{1}{2}hw_{-n}^l$  tons per panel; w being the horizontal pressure of wind per square foot, in tons. And for the bottom lateral system, which alone is affected by the wind pressure against the moving-load, let the uniform moving wind pressure per panel be  $L_1 = \epsilon w_{-n}^l$  tons;  $\epsilon$  being the height of train or other moving-load, in feet.

From (424) we derive the additional weight of top chords due to wind pressure by substituting  $2W_1 = \frac{hwl}{n}$  for (W+L); since, in order to provide for the wind coming either way, we must increase each chord for increased compression, and by putting q for h, and formulating thus,

Weight of top chords due to wind 
$$= \frac{mfl^3hw}{2Qn^2q}(2n^2 + 3n - 2)$$

$$(n \text{ even}),$$

$$= \frac{mfl^3hw}{2Qn^3q}(2n^3 + 3n^2 - 2n - 3)$$

$$(n \text{ odd}).$$
(435)

Similarly, from (425), putting  $(2W_1 + 2L_1) = \frac{wl}{n}(h + 2\ell)$  for (W + L), and q for h,

Weight of bottom chords due to wind 
$$= \frac{mfl^3w(h+2t)}{2Tn^3q}(2n^3-3n^2+22n-24)$$

$$= \frac{mfl^3w(h+2t)}{2Tn^3q}(2n^3-3n^2+22n-21)$$

$$= \frac{mfl^3w(h+2t)}{2Tn^3q}(n \text{ odd}).$$
(436)

And, from (426), we derive the weight of the horizontal struts between the top chords by putting  $W_i = \frac{1}{2}hw_{\overline{n}}^{\underline{l}}$  for  $W_i$ , o for  $L_i$ , q for  $h_i$ ,  $Q_2$  for  $Q_1$ , and adding,

$$\frac{12mfq}{Q_2}W_1n,$$

by reason of the load being applied to the compressed chord, as explained in article 132.

.1

Weight of top horizontal struts due to wind 
$$= \frac{3mfqwlh}{Q_2}(\frac{1}{2}n + 2)$$

$$= \frac{3mfqwlh}{Q_2}\left(\frac{n^2 - 1}{2n} + 2\right)$$

$$= \frac{3mfqwlh}{Q_2}\left(\frac{n^2 - 1}{2n} + 2\right)$$

$$= \frac{3mfqwlh}{Q_2}\left(\frac{n^2 - 1}{2n} + 2\right)$$

$$= \frac{3mfqwlh}{Q_2}\left(\frac{n \text{ odd}}{n \text{ odd}}\right).$$

The floor beams which carry the moving load generally act as the horizontal struts between the loaded chords; and they are usually so large, in comparison with the struts actually required to resist the wind pressure, that we may with little error make no further allowance for these beams acting as horizontal struts than that already suggested in article 124.

But, if it is required, we can find the additional metal to compensate the floor beams for this end pressure by treating each beam as a pillar whose least diameter is its depth, since the longitudinal joists or stringers prevent deflection sideways.

Thus,  $q_1$  being the length, d the depth, of the wrought-iron I floor beams, and S the cross-section due to the total effect of wind pressure, P, in tons, applied longitudinally at the end of a beam, we have, from equation (400),

$$S = \frac{P\left(1 + \frac{(12q_1)^2}{ad^2}\right)}{f_1}$$
 square inches,

to be added to section of each beam, in order to neutralize effect of wind upon the loaded horizontal system of struts.

$$\Sigma S = \frac{1 + \frac{(12q_1)^2}{ad^2}}{f_1} f \Sigma P$$

equals total additional section of I-beams; f being the factor of safety.

Now, in this case,  $\Sigma P$  takes the place of  $\Sigma Z_W$  and  $\Sigma Z_L$ , found by summing the vertical strains, Fig. 112, and used in equation (426), provided we put  $W_i$  for  $W_i$ ,  $L_i$  for  $L_i$ . For, adding  $n(W_i + L_i)$ , since the load is applied on the windward side in the direction of the wind's motion, and subtracting the pressures then upon the end struts, since no struts or I-beams are used on the abutments, will not alter  $\Sigma P$ .

where

$$Q_3 = \frac{f_1}{1 + \frac{(12q_1)^2}{ad^2}}, \quad W_1 = \frac{wlh}{2n}, \quad L_1 = \frac{wls}{n}, \quad m = \frac{5}{18}.$$

l = length, in feet, between centres of end pins.

 $f_{\rm r}$  = numerator of Gordon formula (400).

n =number of panels.

u = constant. (See Table IV.)

h =height of girders, in feet, between centres of chords.

 $q_{\rm r}$  = entire length of floor beam, in feet.

d =depth of beam, in inches.

• = height of train or moving wind-resisting surface.

w = pressure of wind per square foot, in tons.

136. In finding the diagonals of the horizontal systems, top and bottom, due to wind pressure applied on either side, we must plainly make all the diagonals *mains*, and the two in any one panel each equal to the original main tie in that panel.

Using the strain sheet, Fig. 112, as a horizontal system now, putting  $W_1$  for W,  $L_1$  for L, q for h,  $\sin \phi_1 = \frac{nq}{\sqrt{l^2 + n^2q^2}}$  for  $\sin \phi$ ,  $Y_1$  for Y, the strain in any horizontal diagonal tie due



to wind, we have, for the horizontal system between the loaded chords,

Sum of horizontal diagonal strains due to wind
$$\begin{aligned}
&= \sum Y_{1} = \frac{4W_{1}}{2\sin\phi_{1}} \left\{ \frac{n^{2}}{2} - \left( 1 + 3 + 5 + 7 + \cdots \frac{n}{2} \text{ terms} \right) \right\} \\
&+ \frac{4L_{1}}{2n\sin\phi_{1}} \left\{ \frac{n^{3}}{2} - n \left( 1 + 3 + 5 + 7 + \cdots \frac{n}{2} \text{ terms} \right) \right\} \\
&+ 2 \left[ 1 + 3 + 6 + 10 + \cdots \left( \frac{n}{2} - 1 \right) \text{ terms} \right] \right\} \\
&= \frac{W_{1}n^{2}}{2\sin\phi_{1}} + \frac{L_{1}}{12\sin\phi_{1}} (7n^{2} - 4) \\
&= \frac{m_{1}n^{2}}{2\sin\phi_{1}} + \frac{L_{1}}{12\sin\phi_{1}} (7n^{2} - 4)
\end{aligned}$$
(439)

$$= \sum Y_{1} = \frac{4W_{1}}{2\sin\phi_{1}} \left\{ \frac{n-1}{2}n - \left(1+3+5+7+\dots\frac{n-1}{2} \text{ terms}\right) \right\}$$

$$+ \frac{4L_{1}}{2n\sin\phi_{1}} \left\{ \frac{n-1}{2}n^{2} - \left(1+3+5+7+\dots\frac{n-1}{2} \text{ terms}\right) \right\}$$

$$+ 2\left(1+3+6+10\dots\frac{n-3}{2} \text{ terms}\right)$$

$$+ \text{ the } \left(\frac{n-1}{2}\right)^{\text{th}} \text{ term of the series } (1+3+6+10+\dots)$$
for the middle panel \}
$$= \frac{W_{1}(n^{2}-1)}{2\sin\phi_{1}} + \frac{7L_{1}(n^{2}-1)}{12\sin\phi_{1}}$$
(n odd).

Therefore, for horizontal system uniting loaded chords,

Weight of horizontal diagonals due to wind pressure
$$= \frac{mfq}{T\sin^2\phi_1} \left[ 6W_1n^2 + L_1(7n^2 - 4) \right]$$

$$= \frac{mfq(n^2 - 1)}{T\sin^2\phi_1} (6W_1 + 7L_1)$$

$$= \frac{mfq(n^2 - 1)}{T\sin^2\phi_1} (6W_1 + 7L_1)$$

$$= \frac{mfq(n^2 - 1)}{T\sin^2\phi_1} (6W_1 + 7L_1)$$

137. It may be noted here, that, however complete and efficient the horizontal systems are made, they will be unable to maintain the stability of the bridge under the action of wind if the posts and horizontal struts at the ends of the bridge are not sufficient to resist the lateral pressure transmitted to them from these horizontal systems. That is to say, the end framework of the bridge must be, with regard to the wind force, incapable of lateral motion, whether of translation, rotation, or distortion.

The required stability may be secured by making sufficiently large end posts fast to the abutments for light and high structures, and by attaching these end posts to rigid head struts by means of diagonal braces. But as all this excess of weight over the ordinary panel weight rests directly upon the abutments, it does not enter into the formulæ for strains due to the uniform panel pressures, W, L;  $W_1$ ,  $L_2$ .

This excess of weight, however, has an influence on the best values of n and h; and, calling the excess  $E_w$  pounds, we here proceed to formulate its value, and find the conditions of stability.

138. To find the additional strains and weights of the end members of a bridge of two girders of Class IX. required to resist a given wind pressure, let Fig. 114 represent the elevation of the end frame of a through bridge of this class, together with its full moving-load.

Then, according to our previous notation, the total horizontal pressure at A is

$$P_2 = \frac{1}{2}nW_1 = \frac{1}{2}wlh; (442)$$

and at B,

$$P_3 = \frac{1}{2}n(W_1 + L_1) = \frac{1}{4}wl(h + 2\epsilon). \tag{443}$$

The vertical pressure on each abutment is  $\frac{1}{2}n(W+L)$ .

Now, supposing these ends of iron rest upon a plane stone surface, and calling the "co-efficient of friction" for iron upon stone  $\frac{1}{2}$  (see any good treatise on elementary mechanics), we must have, according to the received law of friction,

$$P_2 + P_3 < \frac{1}{4}n(W + L),$$
 (444)

which is the condition that prevents lateral translation along a plane stone surface, BE, Fig. 114.

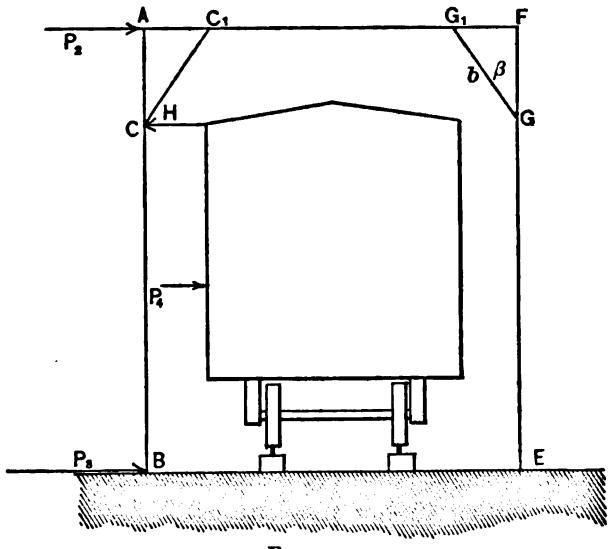


FIG. 114.

For stability against overturning, —

1st, Without live load. Take the moments about E; we need, since AF = q, and FE = h, as above,

$$P_2h < \frac{1}{4}nWq; \tag{445}$$

or, if each end of the girders is tied to the abutment with a force, t,

$$P_{s}h < \frac{1}{4}nWq + qt, \tag{446}$$

the condition that prevents rotation of unloaded bridge about the points of support.

2d, With live load resting upon a beam attached to the girders at the ends B and E, we require the condition

$$P_2h < \frac{n}{4}(W+L)q, \tag{447}$$

girders not tied down; or

$$P_2h < \frac{n}{4}(W+L)q + qt \tag{448}$$

when they are tied with the force t.

But, if the end of the live load rests directly on the abutment, and is not connected with the girders, the condition of stability is

$$P_2h < \frac{1}{4}q[Wn + L(n-1)],$$
 (449)

or

$$P_2h < \frac{1}{4}q[Wn + L(n-1)] + qt.$$
 (450)

3d, For the stability of the load itself against turning on its own points of support, we must have

$$P_4 e < L_2 g; \tag{451}$$

s being the height of moving-load, g the gauge or breadth of base, and  $P_4$  the wind pressure acting upon the part  $L_2$  of this load.

The strains developed in the frame BAFE will be greatest when the end posts cannot move laterally at the bottom nor

are truly fixed, so that, when under wind pressure, the tangents to their elastic curves, at the bases, will be vertical.

Let b = length of brace,  $GG_{i}$ , in feet, and  $\beta = \text{the angle it}$  makes with the vertical post; then—

Shearing-strain at any cross-section of BC or GE is

$$S = \frac{1}{2}P_2. \tag{452}$$

Moment at any point, x, above B or E, is equal to

$$\frac{1}{2}P_2x = M = \frac{1}{2}P_2(h - b\cos\beta)$$
, at C or G. (453)

Taking moments about A for the post,

$$\frac{1}{2}P_2h = Hb\cos\beta,$$

$$\therefore H = \frac{P_2 h}{2b \cos \beta}, \tag{454}$$

which is the horizontal component of the brace strain D.

$$\therefore D = \frac{H}{\sin \beta} = \frac{P_2 h}{2b \sin \beta \cos \beta} \tag{455}$$

in tension or compression.

Moment at any point between  $C_1$  and  $G_2$ ,

$$M = \frac{1}{2}P_2h. \tag{456}$$

To enable each end post to resist this additional moment, (453), it would require R = Bacd as the moment of the internal stresses on the added material, if it be added to the outsides of the post, at the distance  $\frac{1}{2}d$ , in inches, from the neutral axis; B being the ultimate bending unit strength of section, a =width of post, in inches, and c =the uniform thickness of additional iron on each side due to the greatest moment at C or G.

Hence

$$\frac{12}{2}P_2(h-b\cos\beta)=\frac{Bacd}{f},$$

Whole thickness of added iron 
$$= 2c = \frac{12P_2(h - b\cos\beta)f}{aBd}$$
 inches.

Cross-section of added 
$$= 2ac = \frac{12P_2(h - b\cos\beta)f}{Bd}$$
 square inches.

Volume of 4 posts due 
$$= \frac{4 \times 12^{2} P_{2}(h - b \cos \beta) fh}{Bd}$$
 cubic inches.

Weight to be added to 4 posts at end due wind bending,
$$= \frac{12^2 m f w l h^2 (h - b \cos \beta)}{Bd}$$
 pounds. (457)

Similarly, from (456), for the two top horizontal end struts, of length q feet, and depth  $d_i$  inches,

Weight to be added to 2 end struts to resist bending from wind force 
$$= \frac{2 \times 12^{2} mf P_{2}qh}{Bd_{1}}$$
$$= \frac{12^{2} mf wlh^{2}q}{2Bd_{1}}$$
pounds. 
$$(458)$$

If  $d_2$  is the least diameter, in inches, of a brace of length b feet, with fixed ends, to resist the longitudinal pressure D, (455), then, by the Gordon formula, (400), we have

Cross-section of brace, 
$$S = \frac{fD\left\{1 + \frac{(12b)^2}{3000d_2^2}\right\}}{f_1}$$
 square inches;

f, as before, being the factor of safety, and f, the numerator of Gordon formula.

$$\therefore \text{ Volume of 4 braces} = 48bS = \frac{48fD\left\{1 + \frac{(12b)^2}{3000d_2^2}\right\}b}{f_1} \text{ cubic inches.}$$

Weight of 4 braces = 
$$\frac{24mfP_2h\left\{1 + \frac{(12b)^2}{3000d_2^2}\right\}}{f_1 \sin \beta \cos \beta} = \frac{6mfwlh^2\left\{1 + \frac{(12b)^2}{3000d_2^2}\right\}}{f_1 \sin \beta \cos \beta} \text{ pounds.}$$
(459)

$$E_{w} = 12^{2} m f l h^{2} w \left\{ \frac{h - b \cos \beta}{B d} + \frac{q}{2B d_{1}} + \frac{1 + \frac{(12b)^{2}}{3000 d_{2}^{2}}}{24 f_{1} \sin \beta \cos \beta} \right\}, \quad (460)$$

which is the excess of weight, in pounds, of wrought-iron, on the two abutments, due to wind pressure, and not affecting the uniform panel weight of bridge, W.

139. In the preceding investigation, we have assumed that the entire effort of the wind to distort the rectangular cross-section of the bridge is to be resisted by the two end frames alone.

Instead of this provision, however, we may fix firmly each horizontal strut of the unsupported lateral system throughout the bridge to the ends of the posts abutting upon it, and thus transfer the whole wind pressure to that lateral system which is between the chords resting upon the supports. The same transfer would also be accomplished should we connect the posts rigidly to the other, or supported, lateral struts. This procedure would enable us to dispense with the horizontal diagonals of the unsupported system but for the necessity of retaining them to keep the chords they connect from deflecting horizontally.

In this case, of course, the horizontal diagonals and struts of the supported system will have twice as great horizontal pressure to resist as in the former case, and (438) and (441) must be multiplied by 2.

The horizontal struts of the unsupported system must be

able to resist, in a vertical direction, the bending-moment found by (456), when for  $P_2$  we put  $W_1$ , or its value  $\frac{1}{2}wh_n^l$ , giving

$$M = \frac{1}{4}wh^{2}\frac{l}{n} = \frac{Bacd_{2}}{12f}$$
 (461)

for each strut;  $d_2$  being the depth of horizontal strut, in inches, a the width, and c the thickness of each of two plates of iron added at the distance  $\frac{1}{2}d_2$  from the neutral axis of the strut.  $B \div f =$  allowed bending unit strain, in tons, per square inch, since w is in tons. Then the weight of all these horizontal struts due to the bending-moment, (461), must be the same as in (458) if we put  $d_2$  for  $d_1$ , and regard the two extreme struts as one, since each sustains but half a panel pressure.

At the same time, these horizontal struts must resist, in the direction of their least diameters, the bending-moment due to the longitudinal strain brought upon them by the attached horizontal diagonals in adjusting the bridge.

Now these horizontal diagonals, between unsupported chords, may be of uniform size, having a cross-section S, (say) of not less than about I square inch. Then, if the allowed unit strain upon them is  $T \div f$ , and if their inclination to the plane of the girder is  $\phi_{i}$ , we have the longitudinal pressure of two diagonals, from adjustment, to be provided for, equal to

$$P_a = \frac{2TS}{f} \sin \phi_1 = \frac{2TS}{f} \sqrt{\frac{1}{1 + \frac{l^2}{n^2 q^2}}}.$$
 (462)

And if  $\frac{Q_2}{f} = \frac{f_1}{f\left(1 + \frac{(12q)^2}{3000d_3^2}\right)}$  = the allowed pressure per square

inch upon a strut,  $Q_2$ ,  $P_4$ ,  $f_1$ , and T being of the same denom-

ination, and  $f_i$  = numerator of Gordon formula, f = factor of safety, then

$$P_a + \frac{Q_2}{f} = \frac{2TS\sin\phi_1}{Q_2} = S_1,$$
 (463)

which is the cross-section of the strut due to the end pressure  $P_a$ .

Hence, from (461) and (463), —

Total section of a horizontal strut between the unsupported chords is, in square inches,

$$2ac + S_1 = \frac{6wfh^2l}{nBd_2} + \frac{2TS\sin\phi_1}{Q_2}.$$
 (464)

And the weight of n horizontal struts between the unsupported chords, to resist the adjustment and distortion strains, is

$$12mnq(2ac + S_1) = \frac{72mfwlh^2q}{Bd_2} + \frac{24TS\sin\phi_1 mqn}{Q_2}, \quad (465)$$

in pounds, where n is used instead of (n + 1), since the two extreme struts suffer only the strain due to any one of the others.

For the additional iron required in the posts to resist distortion by the wind, we have, from (453), by putting  $\frac{whl}{2n}$  for  $P_2$ , and taking the moment at the centre of post where  $x = \frac{1}{2}h$ ,

$$M = \frac{12wlh^2}{2 \times 4n} = \frac{Bacd}{f} \text{ inch-tons}$$
 (466)

as the bending-moment allowed at the weakest part of the post, each end post having but  $\frac{1}{2}M$  instead of M. Therefore

Whole thickness of added iron for 
$$I$$
  $= 2c = \frac{3wflh^2}{anBd}$  inches.

Cross-section to be added to each  $= 2ac = \frac{3wflh^2}{nBd}$  square inches.

Weight to be added to 
$$2\pi$$
 posts to resist distortion of rectangular cross-section of bridge 
$$= \frac{12^2 m fwlh^3}{2Bd}$$
 pounds. (467)

Finally, the weight of 2n wrought-iron braces for this case also is given by (459); and the cross-section of one brace is

$$S = \frac{fD\left\{1 + \frac{(12b)^2}{3000d_2^2}\right\}}{f_1} = \frac{wflh^2\left\{1 + \frac{(12b)^2}{3000d_2^2}\right\}}{4nf_1b\sin\beta\cos\beta}, \quad (468)$$

since D in (455) now becomes

$$\frac{wh^2l}{4nb\sin\beta\cos\beta}.$$

140. We will now exemplify the method of article 138, which provides, in the end frames alone, the means of resisting the distorting influence of the wind.

Example. — To find the best number of panels, n, and the best height, h, for the two wrought-iron girders of a highway "through" bridge of 100 feet span = l, and 18 feet wide between centres of chords = q, single system of Class IX., Pratt Truss, under a uniform rolling load of 1 ton = 2,000 pounds per running foot, in addition to the weight of bridge. Also, to find the weight, nW, of the bridge corresponding to the best values of n and h, using 4 as the factor of safety for iron, and 10 for wood, and taking account of wind pressure.

Let us compute for n = 5, 6, 7, 8, 9, 10, 11, 12, in succession, as explained in article 134, retaining h and W in all the expressions for weight.

1st, The floor of pine, called 50 pounds per cubic foot.

Thickness  $t = \frac{2.5}{12}$  foot.

Width q' = 17.5 feet.

Length l = 100 feet.

Weight of floor  $F = \frac{2.5}{12} \times 17.5 \times 100 \times 50 = 18229$  pounds.

2d, The joists of pine at 50 pounds per cubic foot.

g = 2 feet between centres.

B = 7,000 pounds per square inch = ultimate resistance to cross-breaking.

f = 10, factor of safety for pine.

 $l \div n =$  panel length of joist, in feet.

 $d = b^2 = \text{depth of joist, in inches, by (431)}.$ 

Then, by (432), we have

Thickness of a joist,  $\delta$  =  $\left\{\frac{9 \times 10 \times 2 \times 100}{n^2 \times 17.5 \times 7000} (18229 + 200000)\right\}^{\frac{1}{8}} = \frac{7.96544}{n^{\frac{3}{8}}}$  ins.; and, from (433),

Weight of joists, 
$$J = \frac{100 \times 17.5 \times 50}{144 \times 2} \left( \frac{7.96544^3}{n^{\frac{11}{8}}} \right) = \frac{153548}{n^{1.2}}$$
 pounds.

3d, The wrought-iron I-beams, n-1 in number, supporting the joists, floor, and moving-load  $L=\frac{l}{n}=\frac{100}{n}$  tons per panel.

Take B = 50,000 pounds, Table II.

Length of beam  $q_1 = 18.5$  feet.

Depth 
$$d = 3.80122 \left\{ \left( \frac{153548}{n^{1.2}} + 218229 \right) \frac{18.5 \times 4}{50000n} \right\}^{\frac{1}{2}}$$
  
= 0.4331885  $\left( \frac{J + 218229}{n} \right)^{\frac{1}{2}}$  inches,

from (412), using the proportions assumed in finding that equation.

By (434),

Weight of I-beams,  $P = 15.46068(n-1) \times \frac{5}{18}$ 

$$\times 18.5 \left( \frac{J + 218229}{n} \times \frac{18.5 \times 4}{50000} \right)^{\text{F}}$$

= 1.031824
$$(n-1)\left(\frac{J+218229}{n}\right)^{\frac{3}{2}}$$
 pounds..

4th, The horizontal struts of the top lateral system of this "through" bridge.

In this example of a highway bridge, let us assume, as actual pressure of wind per square foot, the large value 75 pounds; also that the two open girders offer a resisting surface equivalent to  $\frac{3}{5}$  of the surface presented if the bridge were covered, that is, equal to  $\frac{3}{5}hl$ . Then the whole wind force to be resisted is  $\frac{3}{5} \times 75hl = 45hl$  pounds.

Wind force per running foot = 45h pounds.

Wind force per square foot  $= w = \frac{45}{2000} = 0.0225$  ton.

Although this wind force is actually applied to both girders, we shall regard it as distributed equally to the panel points of the two windward chords, no account being here taken of the action of wind on passing carriages.

Suppose the top horizontal struts to be I-beams, the square of whose least radius of gyration is  $r^2 = 0.5$  inch, which corresponds to a six-inch beam of ordinary make. Then, using equation (385), and calling C = 40,000, E = 27,300,000, we have, in (437),

$$Q_2 = \frac{20}{1 + \frac{40000 \times 210^2}{4\pi^2 \times 27300000 \times 0.5}} = 4.680056 \text{ tons};$$

and (437) gives

Weight of top horizontal struts due to wind
$$= U = \frac{3 \times 5 \times 4 \times 18 \times 0.0225 \times 100h}{18 \times 4.680056} \left(\frac{n}{2} + 2\right)$$

$$= 28.84581h \left(\frac{n}{2} + 2\right) (n \text{ even}),$$

$$= 28.84581h \left(\frac{n^2 - 1}{2n} + 2\right) (n \text{ odd}).$$

5th, The horizontal diagonals, top and bottom. From (441), where now  $L_1 = 0$ , since we take no account here of wind against live load on this highway bridge, we have, making T = 24 tons, q = 18 feet,  $m = \frac{5}{18}$  for wrought-iron (as above),  $W_1 = \frac{hwl}{2n}$ ,

Weight of horizon-  
tal diagonals, top  
and bottom, 
$$= X = \frac{2 \times 5 \times 4 \times 18 \times 6}{18 \times 24 \sin^2 \phi_1} W_1 \begin{cases} n^2 & (n \text{ even}), \\ (n^2 - 1) & (n \text{ odd}), \end{cases}$$
where 
$$\frac{1}{\sin^2 \phi_1} = 1 + \frac{l^2}{n^2 q^2} = 1 + \frac{10000}{18^2 n^2}.$$

6th, Let the residual weight, Y, be 1,000 pounds for all values of n.

7th, The additional weight of iron needed in the I-beams, by reason of their acting as horizontal struts for the wind pressure on lower chord, is found by (438), after computing d as already formulated for the floor beams. Here  $W_1 = \frac{hwl}{2n}$  tons;  $L_1 = 0$ ;  $f_1 = 18$  tons;  $q_1 = 18.5$  feet; a = 750, since the ends are not fixed. Hence, from (438), where  $Q_3 = \frac{18}{1 + \frac{222^2}{750d^2}}$ 

Weight to be added to floor beams due to wind
$$=P' = \frac{3 \times 5 \times 4 \times 18.5 \times 0.0225 \times 100 \left(1 + \frac{222^2}{750d^2}\right)h}{18 \times 18 \times 2}$$

$$\left\{\begin{array}{c} \times n \ (n \text{ even}), \\ \times \frac{n^2 - 1}{n} \ (n \text{ odd}); \end{array}\right.$$

$$P' = 3.8541666h \left(1 + \frac{65.712}{d^2}\right) \left\{\begin{array}{c} \times n \ (n \text{ even}), \\ \times \frac{n^2 - 1}{n} \ (n \text{ odd}). \end{array}\right.$$

Collecting the terms of K thus found, we have, in terms of h,—

Weights of the Components of K, in Pounds.

я	5	6	7	8
Floor Joists I-Beams Do. wind Hor. struts . Hor. diags Residual	18229.0000	18229.0000	18229.0000	18229.0000
	22258.0000	17884.0000	14864.0000	12663.0000
	5459.0000	5969.0000	6408.0000,	6796.0000
	23.3984h	30.1245/k	35.37011	42.3086/2
	126.9216h	144.2291/k	156.59141	173.0749/2
	120.6662h	125.3704/k	125.73361	133.4025/2
	1000.0000	1000.0000	1000.0000	1000.0000
	46946.0000	43082.0000	40501.0000	38688.0000
	+270.9862h	+299.7240/k	+317.69511	+348.7860/2
æ	9	10	11	12
Floor Joists I-Beams Do. wind Hor. struts . Hor. diags Residual	18229.0000	18229.0000	18229.0000	18229.0000
	10994.0000	9688.0000	8641.0000	7784.0000
	7146.0000	7465.0000	7760.0000	8035.0000
	48.1181/1	55.3312h	61.6226/k	69.1289/2
	185.8950/1	201.9207h	215.0294/k	230.7665/2
	138.1040/1	147.2220h	154.0325/k	163.9358/2
	1000.0000	1000.0000	1000.0000	1000.0000
	37369.0000	36382.0000	35630.0000	35048.0000
	+372.1171/1	+404.4739h	+430.6845/k	+463.8312/2

8th, The top chords, of 2 channels and 2 plates of wroughtiron.

In each panel let the ratio of chord's length to least diameter be 15.

Then, in (424),

$$Q = \frac{18}{1 + \frac{15^2}{3000}} = 16.7442 \text{ tons,}$$

by (400).

$$L=\frac{l}{n}=\frac{100}{n}$$
 tons.

Weight of top chords due to vertical pressures, in pounds,
$$\begin{cases}
\frac{5 \times 4 \times 100^{2}}{2 \times 18 \times 16.7442h} \left(W + \frac{100}{n}\right) \\
\times \frac{2n^{2} + 3n - 2}{n} (n \text{ even}), \\
\times \frac{2n^{3} + 3n^{2} - 2n - 3}{n^{2}} (n \text{ odd}).
\end{cases}$$

And, from (435),

Weight of top chords due to wind, in pounds, 
$$= \frac{5 \times 4 \times 100^3 \times 0.0225 h}{2 \times 18 \times 16.7442 \times 18}$$

$$\begin{cases} \times \frac{2n^2 + 3n - 2}{n^2} \ (n \text{ even}), \\ \times \frac{2n^3 + 3n^2 - 2n - 3}{n^3} \ (n \text{ odd}). \end{cases}$$

9th, The bottom chords, of flat links or I-bars. From (425),

And, from (436), a being zero,

10th, The verticals. Take ratio of length to least diameter 30; then, in (426),

$$Q_{1} = \frac{18}{1 + \frac{30^{2}}{750}} = 8.18$$

if the ends are not fixed, and we have

Weight of verticals, in pounds,

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Whn^{2} + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818} \left(\frac{7n^{2} + 3n - 10}{n}\right)$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^{2} - 1) + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^{2} - 1) + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^{2} - 1) + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^{2} - 1) + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^{2} - 1) + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^{2} - 1) + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^{2} - 1) + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^{2} - 1) + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

11th, The girder diagonals, by (428).

$$\frac{1}{\sin^{2}\phi} = 1 + \frac{l^{2}}{n^{2}h^{2}}.$$
Weight of girder diagonals, in pounds,
$$= \frac{4 \times 5 \times 4 \times 100h}{18 \times 24 \sin^{2}\phi} \left(\frac{n^{2}-1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^{2}\phi} Wn^{2}$$

$$(n \text{ even}),$$

$$= \frac{4 \times 5 \times 4 \times 100h}{18 \times 24 \sin^{2}\phi} \left(\frac{n^{2}-1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^{2}\phi} W(n^{2}-1)$$

$$(n \text{ odd}).$$

Computing for the different values of n, collec arranging, we have, including the values of K above, -

## Weights in Pounds, W in Tons, h in Feet.

Live load =  $\kappa L = 100$  tons, l = 100 feet.

==			_			_=: ==:			
# 5	Top chords . \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \	4140.742	W	-	WA.	80815	Ĭ.	-	서
	Bottom chords Load . Wind .	2441-441		-		48889		-	Н
- 1	Verticals	[		9.77778			1	[	H
	Diagonals	1333-333	_	3-33333		35555			П
	К.	-333-333	•	a-33333 —		22222		46946	Н
	2000m W ==	7918.519	—	+13.11111		+167259	-	+46946	H
		, ,,,			<u> </u>	,,,-	Ļ	1.4.24.	<u> </u>
	(Total	4866.256	ľ	_		81104			$  \  $
6	Top chords . Load . Wind .	4000.230		_	١,	01104			Н
		2777.778				48ag6		_	П
	Bottom chords { Load , Wind .	-/////		_		-		-	
	Verticals			14.66667		_	1	-	1 I
	Diagonals	1388.889		5,00000		30007		_	
	<i>R</i> . ,	-		-		-		4308a	Ш
	9000A W =s	9032 923		+19.66667		+157407		+43082	
				<u> </u>		1	ï		ΠĪ
7	Top chords , {Load , Wind ,	5525.323		-		78933	Ι,	-	Н
•		-		-		-		-	Ш
	Bottom chords   Lord . Wind .	3174.604		-		45351	1	-	11
		-		-		-	1	-	П
'	Verticals	, <b>-</b> 1		19.55556		-		-	П
	Diagonals	1360.544		6.66666		25915		-	П
	<i>K</i>	-		-		-	_	40501	]_[.
	2000# W =	10060,471	i	+ 26,22222		+150199		+40501	$\  \ $
				<u> </u>		_ <del></del>		1	T
8	Top chords . Losd . Wind .	6221,067		-		77763	ŀ	-	11
		- [		-		-		-	iΙ
	Bottom chords   Lond .   Wind .	3559.007		-		44488		-	Ш
	Verticals			86.07408		_		_	П
	Diagonala	1388.889		6.8888c		92787			
į	<i>K</i>	-3,9				,-,		38688	ļ
ľ	2000ж₩ «	11168.983	_	+34.96297		+145038		+38688	

## Weights in Pounds, W in Tons, h in Feet.

Live Load =  $\pi L = 100$  tons, l = 100 feet.

==							<del>,</del>		_		<del>=</del>
<b>%</b>	Top chords . \ Load . \ Wind .	6881.576	Wk	-	Wh	76462	1 1	_	ħ°		Ā
		_		-		-		-	1	95-5774	1
	Bottom chords \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \	3978.051 -		- -		4420I -		-		- 55.2507	
	Verticals			32.59260		_		-		402.3777	1
	Diagonals	1371.742		11.11111		20322		-		164.6090	
	<i>K</i>	-3,-0,4-		_		_	1	37369		372.1171	}
ł	2000n W =	12231.369		+43.70371		+140985	-	+37369	-	+1089.9319	+
		3 0 7					<u></u>		1		<u></u>
				1	1		$\overline{1}$		1	1	T
10	Top chords $.$ $\left\{ egin{array}{ll}  ext{Load} \\  ext{Wind} \\ . \end{array} \right.$	7564.819		_		75648		-		-	
	Wind.	-		-		-	1	-	ĺ	94.5602	
	Bottom chords { Load . Wind .	4388.889		-		43889	1	-	1	-	1
	Wind.	-		_	Ī	_	1	-	Į	54.8611	
	Verticals	-	l	40.74075	ĺ	_		-	l	488.8889	
	Diagonals	1388.889		13.88889		18333		-		183.3333	1
	K	-				-		36382		404-4739	
	2000#W=	13342.597		+54.62964		+137870		+36382		+1226.1174	
	(Toad.	8226.202		<u> </u>		74784	Ī		$\overline{\Gamma}$		Ī
11	Top chords . \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \					/4/04	1	_		02.4506	
		4820.936				40905	1			93.4796	
	Bottom chords   Load . Wind .	4020.930				43827	1				
		_	1	0.0000	•	-		-	1	54.7834	
	Verticals	_		48.88888	ł	-		-	1	498.3164	
	Diagonals	1377.410		16.66667		16696	ł	-	1	202,0202	
	K	-		-		-	1	35630	1	430.6845	1
	<b>2000л</b> W =	14424.548		+65.55555		+135307		+35630		+1279.2841	
			Ī	1	1			1	Ī	Ī	〒
12	Top chords . \ Load .	8903.037		-	i	74192		•	1	-	1
1.5	Wind.	-		-		l –	1	-	I	92.7400	1
	Power short Load .	5246.912		-	ļ	43724		-	1	_	1
	Bottom chords   Load . Wind .	`	Ì	-	<u> </u>	_		_	1	54.6553	1
	Verticals	_	1	58.66667	1	_	1	_		585.0823	
	Diagonals	1388.889	{	20.00000		15325		_	1	220.6790	
	K.	-		-		-33-3		35048		463.8312	
	2000nW =	15538.838		+78.66667		+133241	_	+35048	-	+1416.9878	1
			<u></u>				<u> </u>		<u>-</u>	<del>`</del>	<u> </u>

Multiplying each of these eight equations by  $\frac{h}{20000}$ , we find the uniform panel weight of bridge, W, in terms of h; thus:

$$n = 5, W = \frac{8.36295 + 2.3473h + 0.03665487h^{2}}{-0.3959259 + 0.5h - 0.000655555h^{2}}.$$

$$n = 6, W = \frac{7.87035 + 2.1541h + 0.04306191h^{2}}{-0.4516461 + 0.6h - 0.000983333h^{2}}.$$

$$n = 7, W = \frac{7.50995 + 2.02505h + 0.04530032h^{2}}{-0.50302355 + 0.7h - 0.0013111111h^{2}}.$$

$$n = 8, W = \frac{7.2519 + 1.9344h + 0.05197814h^{2}}{-0.55844915 + 0.8h - 0.0017481485h^{2}}.$$

$$n = 9, W = \frac{7.04925 + 1.86845h + 0.05449659h^{2}}{-0.6115684 + 0.9h - 0.002185185h^{2}}.$$

$$n = 10, W = \frac{6.8935 + 1.8191h + 0.06130587h^{2}}{-0.6671298 + h - 0.002731482h^{2}}.$$

$$n = 11, W = \frac{6.76535 + 1.7815h + 0.06396421h^{2}}{-0.7212274 + 1.1h - 0.0032777777h^{2}}.$$

$$n = 12, W = \frac{6.66205 + 1.7524h + 0.07084939h^{2}}{-0.7769419 + 1.2h - 0.00393333h^{2}}.$$

In differentiating these and similar expressions for W, it will be convenient to have a typical form or mode of operation. Let

$$W = \frac{a + bh + ch^2}{a_1 + b_1h + c_1h^2} \tag{469}$$

be a type of these equations; then, after putting  $\frac{dW}{dh} = 0$ , and reducing, we have the equation

$$0 = ab_1 - a_1b + (ac_1 - a_1c)(2h) + (bc_1 - b_1c)h^2, \quad (470)$$

from which h is easily found: and there is no need, in these cases, of taking the second differential to ascertain whether the positive value of h, to be found from (470), renders W a maximum or a minimum; for the substitution of a member a little less or a little greater than the positive value of h so found will at once serve to verify the work, and show W to be a minimum in (469) when h takes the value given by (470).

Taking the case where n = 9, and using logarithms, we may solve thus:—

# = 9.	Logs.	Log Co-efficients.	Co-efficients.	Equation (470).
$a = 7.049250000$ $a_1 = -0.611568400$ $b = 1.868450000$ $b_1 = 0.9000000000$ $c = 0.054496590$ $c_1 = -0.002185185$	0.8481429 9.7864451 0.2714815 9.9542425 8.7363693 7.3394882	$\log ab_1 = 0.8023854$ $\log a_1b = 0.0579266$ $\log bc_1 = 7.6109697$ $\log b_1c = 8.6906118$ $\log ca_1 = 8.5228144$ $\log c_1a = 8.1876311$	$ab_1 = 6.3443300$ $-a_1b = 1.1426900$ $bc_1 = -0.0040829$ $-b_1c = -0.0490469$ $-ca_1 = +0.0333284$ $c_1a = -0.0154039$	- 7.4870200 - = 0.0531298½2 - -0.0179245(2½)

	0.0531 <b>2</b> 98 <b>h</b> ² —	0.0179245(2h)	=	7.48702 <b>00</b>
log co-efficients,	8.7253382	8.2534471		0.8743090
log quotients,		9.5281089		2.1489708
•	h <sup>2</sup> —	0.33737(21)	=	140.9194000
2 log 0.33737,		9.0562178	$= \log$	+0.1138000
_		2.1493213	$= \log$	141.0332000
log (141.0332) <sup>1</sup>		1.0746607	$= \log i$	±11.8757400
½ co-efficient of h,			_	+0.3373700

When W is a minimum, h = 12.21311 feet.

```
1.0868263
\log h,
                                      7.049250
                           bh =
log bh,
            1.3583078
                                     22.819590
\log h^2,
            2.1736526
                                      8.128720 \quad 37.997560 \quad 1.5797557 = \log \text{num}.
\log ch^2,
                           ch^2 =
            0.9100219
\log c_1 h^2,
           9.5131408
                           c_1h^2 = -0.325942
                             a_1 = -0.611568
                   b_1h = 0.9h = 10.991799 10.054289 1.0023513 = log denom.
                nW = 34.013151 \text{ tons.} W = 3.779239 \text{ 0.5774044} = \log W.
```

Computing h and Wn for the other values of n, we find them, when Wn is a minimum, as follows:—

No. of Panels,	Panel Length, I + n feet,	Best Height in Feet, A.	Ratio of Length to Height, l+h.	Inclination of Diagonals to Horison,	Minimum Bridge Weight, #W Tons.	Ratio of Dead to Live Load,	Ratio of Dead to Total Load.
5 6	30 26 14 <sup>3</sup>	16.50041 14.71481 13.89555	6,0604 6,7959 7,1966	39° 31′ 26″ 41° 26′ 30″ 44° 12′ 24″	37-177990 35-930016 34-642979	0.37178 0.35930 0.34643	0,27100 0,26433 0,25730
8	199	12.74032	7.8491	45° 32′ 44″	34.509872	0.34510	0.25656
9	219	12,21311	8.1879	47 42 18"	34.013151	0.34013	0.25380
10	10	11.39809	8.7734	48° 44′ 18″	34.302300	0.34309	0.25541
11	911	11.02062	9.0739	50" 28" 51"	34.156870	0.34157	0.85460
12	8 1	10.40797	9.6080	51° 19′ 1″	34.6350ta	0.34635	0.95725

Span = l = 100 Feet. Uniform Live Load = nL = 100 Tons.

Hence 34.013151 tons is the least of these least weights, or the minimum minimorum.

By observing the first differences of the values of nW, we nay perceive, that, in addition to the fact that the odd number n = 0 gives the lowest value of nW, the odd number n = 11 gives a lower value of nW than either of the even numbers, 10 and 12, adjacent to it, and that n = 7 renders nW nearly as small as n = 8. We may, therefore, almost infer from these eight cases, that, when near the best value of n, an odd number of panels is preferable to an even number. And this conclusion narmonizes with the fact, that, in case of an odd number of panels, there is no weight applied at the centre of span as there is when n is even. We may further observe that the difference between the greatest and least values of nW in these eight cases is only 3.164839 tons, provided the best values of n are used; but, if other values of n are employed, n departs more widely from its least value.

Also in the present case, when nW is least, the inclination,  $\phi$ , of the girder diagonals to the horizon is about  $2\frac{n}{4}$  degrees above 45 degrees; and from this point  $\phi$  increases or decreases with n if the best value is given to h.

The best ratio of length to height of girder for this span and load is 8.1879; and, near the best simultaneous values of n and h, we have approximately

$$h = \frac{l}{n}.\tag{471}$$

12th, Let us now find the value of  $E_{w}$ , equation (460), the quantity of wrought-iron to be added to the end framework to resist wind force tending to produce distortion, assuming that the bridge is so fixed to the abutments that neither sliding nor overturning can take place.

In equation (460), take b = 4 feet = length of brace,  $d_2 = 6$  inches,  $\beta = 45$  degrees = inclination of brace to post.

$$\therefore \sin \beta = \cos \beta = 0.70711, \quad b \cos \beta = 2.82844 \text{ feet.}$$

Take d = 12 inches, width of end post to resist bending.

 $d_1 = 12$  inches, depth of end horizontal strut.

 $f_1 = 18$  tons.

B = 25 tons.

q = 18 feet.

 $m = \frac{5}{18}$  pound.

f = 4

l = 100.

w = 0.0225 ton.

Then, computing  $E_{w}$  for the eight values of h already found, we obtain, from (460) and from the table just given, the following results:—

W in Tons, & in Feet.

No. of Panels, n.	5	6	7	8
Height, h	16.500	14.715	13.896	12.740
n times panel weight, $nW$	37.178	35.927	34.643	34.510
Added iron, $E_{w}$ tons	3-935	2.898	2.489	1.980
Weight of bridge, $nW + E_{w}$ .	41.113	38.825	37.132	36.490
Weight of wood	20.244	18.057	16.547	15-446
Weight of iron	20.869	20.768		21.044
Cost of iron, at \$150	\$3130 35	\$3115 20	\$3087 75	\$3156 60
Cost of wood, at \$15				231 69
Cost of bridge		3386 06		3388 29
Excess over least		50 10	-	5° 33
			<del></del>	
No. of Panels, π.	9	10	11	12
Height, h	12.213	11.398	11.021	10.408
Height, h	12.213 34.013	11.398 34.302	11.021 34.157	10.408
Height, h	12.213	11.398 34.302	11.021 34.157	10.408 34.635
Height, $h$	12.213 34.013 1.773 35.786	11.398 34.302 1.480 35.782	11.021 34.157 1.357 35.514	10.408 34.635 1.170 35.805
Height, $h$	12.213 34.013 1.773	11.398 34.302 1.480	11.021 34.157 1.357	10.408 34.635 1.170
Height, $h$	12.213 34.013 1.773 35.786 14.612	11.398 34.302 1.480 35.782 13.959	11.021 34.157 1.357 35.514	10.408 34.635 1.170 35.805
Height, $h$	12.213 34.013 1.773 35.786 14.612 21.174	11.398 34.302 1.480 35.782 13.959 21.823	11.021 34.157 1.357 35.514 13.435 22.079	10.408 34.635 1.170 35.805 13.007 22.798
Height, $h$	12.213 34.013 1.773 35.786 14.612 21.174	11.398 34.302 1.480 35.782 13.959 21.823	11.021 34.157 1.357 35.514 13.435 22.079	10.408 34.635 1.170 35.805 13.007 22.798

Here we see that n = 11 and h = 11.021 are the conditions yielding least total weight of bridge, while the whole cost is a minimum if n = 7, h = 13.896, and  $(l \div n) = 14\frac{2}{7}$ .

Notice that both of these minima of weight and cost correspond to an odd number of panels, and that the excess of cost above the lowest would in all cases more than compensate the

manufacturer for having the best simultaneous values of n and h determined by calculation, as above.

If, however, there would be sufficient head-room, we may, for this span and load, adopt either 8 or 9 panels, giving more iron, less wood, and less total weight, with a small increase of cost. In each of these 8 cases it will be seen the bridge weight is a little more than one-third the uniform moving-load, 100 tons = nL, and that the total dead load is slightly greater than one-fourth the sum of dead and live loads.

141. To exemplify the Method of Article 139, which provides, at Every Post, the Means of resisting the Distorting Influence of the Wind. — Taking the example of article 140, and calling the top horizontal diagonals 1 inch in diameter (that is, 0.7854 square inch cross-section), and weighing 2.654 pounds to the foot, we have

Weight of 
$$2n$$
 horizontal top  $= 2n \times 2.654\sqrt{18^2 + \frac{100^2}{n^2}}$   
=  $5.308\sqrt{324n^2 + 10000}$  pounds,

Weight of bottom horizontal = X,  $\begin{cases} as found in article 140, for both top and bottom. \end{cases}$ 

Strain on a top horizontal strut, from  $\frac{24}{4} = 6$  tons per square inch on two top diagonals, is equal to

$$2 \times 6 \times 0.7854 \sin \phi_r = 9.4248 \sin \phi_r \cos$$
.

Now we already have the breaking inch strain on top horizontal struts = 4.680056 tons, and

$$\sin \phi_1 = \sqrt{\frac{1}{1 + \frac{l^2}{n^2 q^2}}} = nq \sqrt{\frac{1}{n^2 q^2 + l^2}} = \sqrt{\frac{1}{1 + \frac{10000}{324n^2}}}.$$

Therefore, in square inches,

Cross-section of a top strut to resist initial strain on diagonals 
$$= \frac{9.4248}{1.170014} \sin \phi_1 = 8.0553 \sqrt{\frac{1}{1 + \frac{10000}{324\pi^2}}} = S_1.$$

And, from (461),

Cross-section of a top strut to resist distorting force of wind 
$$= 2ac = \frac{6wflh^2}{Bd_2n} = 0.30\frac{6}{7}\frac{h^2}{n} \text{ square inches}$$

if w = 0.0225 ton, f = 4, l = 100, B = 25 tons, and  $d_2 = 7$  inches.

From (465), since  $12mqn = 12 \times \frac{5}{18} \times 18n = 60n$ ,

Weight of 
$$n$$
 top horizontal struts  $= 18.5143h^2 + 483.318n^2\sqrt{\frac{1}{n^2 + 30.8642}}$  pounds  $= 1851.428\frac{h}{n} + 483.318n^2\sqrt{\frac{1}{n^2 + 30.8642}}$  pounds,

approximately, by reason of (471), to avoid the second power of h, for convenience.

Weight to be added to floor beams due to wind = two times P',

as already given.

In (467) take d = 8 inches; then

Weight to be added to all posts 
$$= \frac{12^2 \times 5 \times 4 \times 0.0225 \times 100}{2 \times 18 \times 25 \times 8} h^3$$
$$= \frac{9000h}{n^2} \text{ pounds,}$$

by (471).

In the previous case the quantity of iron of uniform thickness to be added to each post is that due to the greatest moment given by equation (453). It is plain from that equation that the added iron may vary in thickness from C, Fig. 114, where it should be greatest, to the bottom, where it may be nothing. Or, without increasing the thickness of the iron, the post may be made broader at top than at bottom, and thus resist the bending-moment whenever this broadening is not accompanied by too great reduction of the thickness of the iron composing the post. In the present case  $x = \frac{1}{2}h$ .

Finally, from (459), calling  $d_2 = 4$  inches,

Weight of all braces 
$$= \frac{6 \times 5 \times 4 \times 0.0225 \times 100}{18 \times 18 \times 0.70711^{2}} \left(1 + \frac{48^{2}}{3000 \times 4^{2}}\right) h^{2}$$

$$= 1.74667 h^{2}$$

$$= 174.667 \frac{h}{n} \text{ pounds,}$$

by (471).

Computing for 8 values of n, we find, —

Weights of the Components of K, in Pounds.

н.	5	6	7	8
Floor	18229.0000	18229.0000	18229.0000	18229.0000
	22258.0000	17884.0000	14864.0000	12663.0000
	5459.0000	5969.0000	6408.0000	6796.0000
	46.7968h	60.2490 <i>h</i>	70.7402h	84.6172 <i>h</i>
	1617.0000	2128.0000	2650.0000	3176.0000
	370.2857h	308.5714 <i>h</i>	264.4898h	231.4286 <i>h</i>
Horizontal diagonals {	714.0000	781.0000	854.0000	931.0000
	120.6662 <i>h</i>	125.3704h	125.7336h	133.4035 <i>h</i>
Braces	34.9333½	29.1111 <i>h</i>	24.9524h	21.8333 <i>h</i>
	1000.0000	1000.0000	1000.0000	1000.0000
	572.6820½	523.3019 <i>h</i>	485.9160h	471.2816 <i>h</i>
	+49277	+45991	+44005	+42795

n.	9	10	11	12
Floor	18229.0000 10994.0000	18229.0000 9688.000p	18229.0000 8641.0000	18229.0000 7784.0000
I floor beams	7146.0000 96.2362 <i>h</i>	7465.0000	7760.0000 123.2452h	8035.0000 138.2578#
Horizontal top struts {	3701.0000	4225.0000	4746.0000	5263.0000
Horizontal diagonals {	205.7143/2	185.1429 <i>h</i> 1093.0000	168.3117 <i>h</i>	154.2857k
Braces	138.1040 <i>h</i> 19.4074 <i>h</i>	147.2220h 17.4667h	154.0325h 15.8788h	163.9358# 14.5555#
Residual $K$	1000.0000 459.4619 <i>h</i>	1000.0000 460.4940 <i>h</i>	1000.0000 461.4682 <i>h</i>	1000.0000 471.0348/£
- }	+42081	+41700	+41553	+41574

Weights of the Components of K, in Pounds. — Concluded.

The strain throughout each top chord due to the initial strain,  $6 \times 0.7854 = 4.7124$  tons on each diagonal between top chords, is

$$4.7124\cos\phi_1$$
 tons,

and the allowed inch strain here is

$$\frac{16.7442}{4} = 4.18605$$
 tons.

Therefore the additional cross-section of iron for both top chords due to initial strain on top diagonals is, in square inches,

$$\frac{2 \times 4.7124}{4.18605} \cos \phi_1 = 2.25148 \cos \phi_1 = \frac{225.148}{\sqrt{324n^2 + 10000}}$$

Additional weight for top chords, pounds, due initial strain on top diagonals 
$$= \frac{12 \times 100 \times 5 \times 225.148}{18\sqrt{324n^2 + 100^2}} = \frac{75049.333}{\sqrt{324n^2 + 10000}}$$

The effect of wind on the bottom chords in this case will be twice what it was in the example of article 140, and may be taken from the table therein given.

Also, the weights of the girder diagonals will be the same as given in that article.

We may expect a heavier bridge this time than was found in the last example, by reason of the initial strain now assumed on the top diagonals, and the smaller values of d for the top struts, the posts, and the braces, in comparison with the values used in the two end frames to resist wind.

Computing weights for the different values of *n*, and collecting results, we have,—

l = 100 Foot-Weights in Pounds, W and L in Tons, A in Feet, mL = 100 Tons.

<b>*</b>	(Toad	4140.742	Wh		WA	82815	Ŀ		4.	_	1,
5	Top chords { Load Initial st.,	4.40./4#	h	_	,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,	-	A	558		-	
	Load .	2444-444		_		48889		-		-	1
ŀ	Bottom chords \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \	-		-		-	l	-		122,2323	ı
	Verticals, total	-		9.77778		-		_		568.5926	
	Girder diagonals	1333-333		3-33333		35555		-		88.889	
1	K	-		-	ļ	-		49277		572.6820	1
	2000n W =	7918.519		+13.11111		+167259		+49835		+1352.3857	
6	Top shords   Load	4866.256			•	81104		-		-	Ī
۱	Top chords { Load Initial st.,	-		-		-		510		-	1
İ	Bottom chords $\left\{ egin{array}{ll} { m Load} & { m Wind} & { m Wind} & { m Const.} \end{array} \right.$	2777.778		-	]	46296	1	_		-	
ı		-		-		-		-		115.7408	
ı	Verticals, total	-		14.66667		-		-		544.2387	
1	Girder diagonals	1388.889		5.00000		30007	1	-		108.0247	
	<i>K</i>	-		-		-		45991	_	523.3019	_ _
	2000n W =	9032.923		+19.66667		+157407		+46501		+1291.3061	
,	Top chords { Load Initial st.,	5525. <b>32</b> 3		-		78933		_		-	Ī
<b>'</b>	Initial st.,	-		-		-	ł	467		-	
	Bottom chords { Load . Wind .	3174.604		-		45351	1	-		-	l
		-		-		-		-		113.3788	
	Verticals, total	<b>-</b>		19.55556		-		_		489.6447	
	Girder diagonals	1360.544	1	6.66666		25915		-		126.9841	
	<i>K</i>			-				44005		485.9160	
	2000nW ==	10060.471		+26.2222		+150199		+44472		+1215.9236	<b> </b>

l = 100 Foot-Weights in Pounds, W and L in Tons, k in Feet,  $\pi L = 100$  Tons.

n	(1,004		W	1	.,,,					!
8	Top chords Load Initial st.,	6221.067	h	-	Wh	77763	上水	0	4°	_
	(Tood	-				00		418		_
	Bottom chords { Load . Wind .	3559.027		_	•	44488		-		_
		_				_		_		111.2196
	Verticals, total Girder diagonals	00 00-		26.07408 8.88889				-		532.7546
	K.	1300.009		0.00004		22787		40704		145.8333
	Λ.,						.	42795		471.2816
	2000# W =	11168.983		+34.96297		+145038		+43213		+1261.0891
	( I and	6881.576				<b>#</b> 5.60	Ī			
9	Top chords { Load Initial st.,	-		_		76462		204	li	_
	-	)	•	_		44201	1	394		_
- 1	Bottom chords   Load . Wind .	39/0.051		<u> </u>		44201		_	1 1	-
	Verticals, total	_		32.59260		_		_		110.5014 513.48 <b>8</b> 8
	Girder diagonals	1271 749		11.11111		90322		_		513.4000 164.6090
	K.	-3/**/**				-		42081	1	459-4619
j					<b> </b>		- -	4-01		
	2000nW =	12231.369	<u> </u>	+43.70371		+140985		+42475		+1248.0611
	m (Load	7564.819		_		75648		-		-
2	Top chords { Load Initial st.,	-		-		7,5040		364		_
	- 1	4388.889	}	_		43889		-		_
	Bottom chords { Load . Wind .	_		-		_	1	-	1	109.7222
	Verticals, total	-	}	40.74075		-		_	1	578.8839
	Girder diagonals	1388.889		13.88889		18333		-		183.3333
	<i>K</i>	•		-	1	-		41700		460.4940
	2000n W =	13342.597	-	+54.62964		+137870		+42064	-	+1332.4384
			<u>:</u>				1			<del></del>
	Top chords Load Initial st.,	8226.202		-		74784	İ	-		-
				-		-		338		-
	Bottom chords { Load . Wind .	4820.936		_		43827		_		-
	·	-		.0.0000		-		-		209.5668
	Verticals, total			48.88888		-		-	{	572.6966
-	Girder diagonals	1377.410		16.66667		16696				902,0208
	<i>K</i>			-			_	41553		461.4682
	2000n W =	14424.548		+65.55555		+135307		+41891		+1345.7518
	m . (Inad	8903.037		_		74192		-		-
2	Top chords \ Load Initial st.,	-7-33/		_		77-9-		315		-
	•	5246.912		_		43724		J-3		-
	Bottom chords \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \	J-7-1 <b>7</b> -2		_		73/-7 -		_		109.3106
į	Verticals, total	-		58.66667		_		_		647.5823
į	Girder diagonals	1388.889		20.00000		15325		_		220.6790
•	K	-		-				4 <sup>1</sup> 574		471.0348
		7550 000					╢		-	
	2000#W =	15530.038		+78.66667		+133241	1	+41889		+1448.6067

Multiplying each of these eight equations by  $\frac{h}{20000}$ , we find the uniform panel weight of bridge, W, in terms of h; thus:

$$n = 5, W = \frac{8.36295 + 2.49175h + 0.06761929h^2}{-0.3959259 + 0.5h - 0.00065555h^2}$$

$$n = 6, W = \frac{7.87035 + 2.32505h + 0.06456531h^2}{-0.4516462 + 0.6h - 0.00098333333h^2}$$

$$n = 7, W = \frac{7.50995 + 2.2236h + 0.06079618h^2}{-0.5030235 + 0.7h - 0.0013111111h^2}.$$

$$n = 8$$
,  $W = \frac{7.2519 + 2.16065h + 0.06305446h^2}{-0.5584492 + 0.8h - 0.001748148h^2}$ 

$$n = 9$$
,  $W = \frac{7.04925 + 2.12375h + 0.06240306h^2}{-0.61156845 + 0.9h - 0.002185186h^2}$ 

$$n = 10$$
,  $W = \frac{6.8935 + 2.1032h + 0.06662192h^2}{-0.66712985 + h - 0.002731482h^2}$ 

$$n = 11$$
,  $W = \frac{6.76535 + 2.09455h + 0.06728759h^2}{-0.7212274 + 1.1h - 0.003277777h^2}$ 

$$n = 12$$
,  $W = \frac{6.66205 + 2.09445h + 0.07243034h^2}{-0.7769419 + 1.2h - 0.00393333h^2}$ 

Differentiating these equations, and putting  $\frac{dW}{dh} = 0$ , we find results as here tabulated; h corresponding to the least value of nW.

Span l = 100 Feet, Uniform Live Load =  $\pi L = 100$  Tons.

Number of Panels, s. •	5	6	7	8
Height in feet, h	12.69087			
Weight of bridge, tons, nW	43.51195	40.91832	38.99626	38.45946
Panel length, $l \div n$	20.00000		I4 <del>†</del>	I 2 1
$l \div h \cdot \cdot \cdot \cdot$	7.87970	8.05800	8.12640	_
Slope of diagonals, $\phi$	32° 23′ 50″	36° 40′ 17″	40° 44′ 28″	43° 19′ 51″
Ratio of dead to live load	0.43510	0.40920	0.39000	. 0.38460
Ratio of dead to total load	0.30320	0.29040	0.28050	0.27780
Weight of bridge per lin. ft., lbs.,	870.00000	818.00000	770.00000	769.00000
Weight of wood, tons	20.22400	18.05700	16.54700	15.44600
Weight of iron, tons	23.28800	22.86100	22.44900	23.01300
Cost of iron, at \$150	\$3493 20	\$3429 15	\$3367 35	\$345I 95
Cost of wood, at \$15	303 36	270 86	248 20	231 69
Cost of bridge	3796 56	3700 OI	3615 55	3683 64
Excess over least	181 01	84 46	0	68 <b>o</b> 9
Cost per linear foot	37 97	37 00	36 16	36 84
Number of Panels, #.	9	10	11	12
Height in feet, h	11.59226	11.06714	0.0	
		11.06714	10.82859	10.37000
Weight of bridge, tons, nW	Ξ,	_	-	J — —
Weight of bridge, tons, $nW$ . Panel length, $l \div n$ .	37.83525	38.08053	38.04843	38.60209
	37.83525	38.08053 10.00000	38.04843 9 <del>11</del>	38.60209 81
Panel length, $l \div n$	37.83525	38.08053 10.00000 9.03580	38.04843 9 <del>11</del> 9.23480	38.60209 81
Panel length, $l \div n$	37.83525 11 <del>1</del> 8.62640	38.08053 10.00000 9.03580 47° 54′ 0″	38.04843 9 <del>11</del> 9.23480 49° 59′ 8″	38.60209 8 <del>1</del> 9.63700 51° 13′ 57″
Panel length, $l \div n$	37.83525 11 <del>1</del> 8.62640 46° 12′ 51″ 0.37840	38.08053 10.00000 9.03580 47° 54′ 0″	38.04843 9 <del>11</del> 9.23480 49° 59′8″ 0.38050	38.60209 81/3 9.63700 51° 13′ 57″ 0.38600
Panel length, $l \div n$	37.83525 11 <del>1</del> 8.62640 46° 12′ 51″ 0.37840	38.08053 10.00000 9.03580 47° 54′ 0″ 0.38080 0.27580	38.04843 9 <del>11</del> 9.23480 49° 59′8″ 0.38050 0.27560	38.60209 81/3 9.63700 51° 13′ 57″ 0.38600
Panel length, $l \div n$	37.83525 1113 8.62640 46° 12′ 51″ 0.37840 0.27450	38.08053 10.00000 9.03580 47° 54′ 0″ 0.38080 0.27580 762.00000	38.04843 911 9.23480 49° 59′8″ 0.38050 0.27560 761.00000	38.60209 81/3 9.63700 51° 13′ 57″ 0.38600 0.27850 772.00000
Panel length, $l \div n$ $l \div h$ Slope of diagonals, $\phi$ Ratio of dead to live load  Ratio of dead to total load  Weight of bridge per lin. ft., lbs.,	37.83525 1113 8.62640 46° 12′ 51″ 0.37840 0.27450 757.00000 14.61200	38.08053 10.00000 9.03580 47° 54′ 0″ 0.38080 0.27580 762.00000 13.95900	38.04843 911 9.23480 49° 59′8″ 0.38050 0.27560 761.00000 13.43500	38.60209 81/2 9.63700 51° 13′ 57″ 0.38600 0.27850 772.00000 13.00700
Panel length, $l \div n$ . $l \div h$ .  Slope of diagonals, $\phi$ .  Ratio of dead to live load.  Ratio of dead to total load.  Weight of bridge per lin. ft., lbs.,  Weight of wood, tons	37.83525 1113 8.62640 46° 12′ 51″ 0.37840 0.27450 757.00000 14.61200	38.08053 10.00000 9.03580 47° 54′ 0″ 0.38080 0.27580 762.00000 13.95900	38.04843 911 9.23480 49° 59′8″ 0.38050 0.27560 761.00000 13.43500	38.60209 81/3 9.63700 51° 13′ 57″ 0.38600 0.27850 772.00000 13.00700
Panel length, ! + n	37.83525 1113 8.62640 46° 12′ 51″ 0.37840 0.27450 757.00000 14.61200 23.22300	38.08053 10.00000 9.03580 47° 54′ 0″ 0.38080 0.27580 762.00000 13.95900 24.12200	38.04843 911 9.23480 49° 59′ 8″ 0.38050 0.27560 761.00000 13.43500 24.61300	38.60209 81/2 9.63700 51° 13′ 57″ 0.38600 0.27850 772.00000 13.00700 25.59500
Panel length, ! ÷ n	37.83525 11 d 8.62640 46° 12′ 51″ 0.37840 0.27450 757.00000 14.61200 23.22300 \$3483 45 219 18	38.08053 10.00000 9.03580 47° 54′ 0″ 0.38080 0.27580 762.00000 13.95900 24.12200 \$3618 30 209 38	38.04843 911 9.23480 49° 59′ 8″ 0.38050 0.27560 761.00000 13.43500 24.61300	38.60209 81/2 9.63700 51° 13′ 57″ 0.38600 0.27850 772.00000 13.00700 25.59500  \$3839 25 195 11
Panel length, ! + n	37.83525 11 d 8.62640 46° 12′ 51″ 0.37840 0.27450 757.00000 14.61200 23.22300 \$3483 45 219 18	38.08053 10.00000 9.03580 47° 54′ 0″ 0.38080 0.27580 762.00000 13.95900 24.12200 \$3618 30 209 38 3827 68	38.04843 911 9.23480 49° 59′ 8″ 0.38050 0.27560 761.00000 13.43500 24.61300 \$3691 95 201 53 3893 48	38.60209 81/2 9.63700 51° 13′ 57″ 0.38600 0.27850 772.00000 13.00700 25.59500  \$3839 25 195 11 4034 36

Here, again, we find least weight, nW = 37.83525 tons, answering to the odd number of panels, 9, and the height, h = 11.59226 feet; while the inclination of diagonals to horizon,  $\phi$ , is about 1½ degrees above 45 degrees.

The least cost, at the rates here assumed, corresponds to 7 panels; it being understood that we have once or twice employed the approximation involved in (471).

142. Again, by the method of article 139, take the same example, except that the uniform live load is now 2 tons to the linear foot, instead of 1 ton, as in article 141.

1st, The floor, as before, weighs

$$F = \frac{2.5}{12} \times 17.5 \times 100 \times 50 = 18229$$
 pounds.

2d, By (432),

Thickness of a joist, 
$$b = \left\{ \frac{9 \times 10 \times 2 \times 100}{n^2 \times 17.5 \times 7000} (18229 + 400000) \right\}^{\frac{1}{6}}$$

$$= \frac{9.07215}{n^{\frac{3}{6}}} \text{ inches.}$$

And, from (433),

Weight of joists, 
$$J = \frac{100 \times 17.5 \times 50}{144 \times 2} \left( \frac{9.07215^3}{n^{1.2}} \right) = \frac{226854}{n^{1.2}}$$
 pounds.

3d, Depth of I floor beams, from (412), as in article 140,

$$d = 3.80122 \left\{ \left( \frac{226854}{n^{1.2}} + 418229 \right) \frac{18.5 \times 4}{50000n} \right\}^{\frac{1}{3}}$$

$$= 0.4331885 \left( \frac{J + 418229}{n} \right)^{\frac{1}{3}} \text{ inches.}$$

By (434),

Weight of I-beams,

$$P = 15.46068(n-1) \times \frac{5}{18} \times 18.5 \left( \frac{J + 418229}{n} \times \frac{18.5 \times 4}{50000} \right)^{\frac{3}{2}}$$

$$= 1.031824(n-1) \left( \frac{J + 418229}{n} \right)^{\frac{3}{2}} \text{ pounds.}$$

4th, Take top horizontal diagonals, each  $1\frac{1}{8}$  inches in diameter. Cross-section = 0.99402 square inches; weight = 3.359 pounds per foot. Then

Weight of 2n top horizontal diagonals

$$= 2n \times 3.359 \sqrt{18^2 + \frac{100^2}{n^2}} = 6.718 \sqrt{324n^2 + 10000}$$
 pounds.

Weight of bottom horizontal diagonals 
$$= X = \frac{2 \times 5 \times 4 \times 18 \times 6}{18 \times 24 \sin^2 \phi_1} W_1 \begin{cases} n^2 & (n \text{ even}), \\ (n^2 - 1) & (n \text{ odd}), \end{cases}$$

as in article 140, for both top and bottom.

$$W_{\rm I} = \frac{hwl}{2n}, \quad \frac{1}{\sin^2 \phi_{\rm I}} = 1 + \frac{10000}{18^2 n^2}.$$

5th, Strain on each top horizontal strut from  $\frac{24}{4} = 6$  tons per square inch on two top diagonals =  $2 \times 6 \times 0.99402$   $\sin \phi_1 = 11.92824 \sin \phi_1$  tons; allowed inch strain on strut =  $\frac{4.680056}{4} = 1.170014$  tons. Therefore

Cross-section of a top strut to resist initial strain on diagonals
$$= \frac{11.92824}{1.170014} \sin \phi_1 = 10.19495 \sqrt{\frac{1}{1 + \frac{10000}{324n^2}}} = S_1.$$

From (461),

Cross-section of a top strut to resist distorting force 
$$\begin{cases} = 2ac = \frac{6wflh^2}{Bd_2n} = 0.30\frac{6h^2}{n} \text{ square inch.} \end{cases}$$

From (465),

Weight of 
$$n$$
 top horizontal struts 
$$= 18.5143h^2 + 611.697n^2\sqrt{\frac{1}{n^2 + 30.8642}}$$
 pounds 
$$= 1851.428\frac{h}{n} + 611.697n^2\sqrt{\frac{1}{n^2 + 30.8642}}$$
 pounds,

by reason of (471).

6th, Weight to be added to floor-beams, due to wind,  $= 2 \times P'$  in article 140, changing d.

7th, Weight to be added to all posts to resist distortion by wind =  $\frac{9000}{n^2}h$  pounds, as before.

8th, Weight of all braces =  $174.667\frac{h}{n}$  pounds, as before. Computing for 8 values of n, we find,—

Weights of Components of K, in Pounds. l = 100 Feet,  $\pi L = 200$  Tons.

æ.	5	6	7	8
Floor Joists I floor beams Do. wind Horizontal top struts  Horizontal diagonals  Braces Residual  K	18229.0000	18229.0000	18229.0000	18229.0000
	32884.0000	26422.0000	21960.0000	18708.0000
	8303.0000	9102.0000	9790.0000	10398.0000
	43.4410/2	55.4297 ½	64.5618/2	76.6677½
	2046.0000	2693.0000	3354.0000	4019.0000
	370.2857/2	308.57 14½	264.4898/2	231.4286½
	904.0000	989.0000	1081.0000	1178.0000
	120.6662/2	125.3704½	125.7336/2	133.4035½
	34.9333/2	29.1111½	24.9524/2	21.8333½
	1200.0000	1200.0000	1200.0000	1200.0000
	569.3262/2	518.4826½	479.7376/2	463.3331½
	+63566	+58635	+55614	+53732

#### MECHANICS OF THE GIRDER.

Weights of Components of K, in Pounds. l = too Feet,  $\pi L = \text{soo Tons}$ .

=	•	10	11	19
r	18229.0000 16243.0000 10944.0000	18229.0000 14313.0000 11443.0000	18229.0000 12767.0000 11903.0000	18229.0000 11501.0000 12331.0000
wind zontal top struts {	86.61644 4684.0000 205.71434	5347.0000 185.1429#	6006.0000 168.3117#	122.3137# 6661.0000 154.2857#
zontal diagonals	1279.0000 138.1040Å 19.4074Å	1383.0000 147.2220/t 17.4667/t	1490.0000 154.03254 15.87884	1599.0000 163.9358# 14.5555#
inal	1200.0000 449.8421 <i>h</i> +52579	448.82124 +51915	1200.0000 447.8397.4 +51595	1200.0000 1 455.0907# +51521

, Taking Q = 16.7442 tons, as in article 140,  $L = \frac{00}{n}$  tons, we now have

ain throughout each top chord due to initial strain of 0.99402 = 5.96412 tons, along each diagonal between ords, is

5.96412 cos φ1 tons.

wed inch pressure on top chords =  $\frac{16.7442}{4}$  = 4.18605 tops.

Additional cross-section of iron for both top chords due to initial strain on top diagonals 
$$= \frac{2 \times 5.96412}{4.18605} \cos \phi_1$$

$$= \frac{284.952}{\sqrt{324n^2 + 10000}}$$
 square inches.

top chords due in-  
itial strain on top  
diagonals, pounds
$$= \frac{12 \times 100 \times 5 \times 284.952}{18\sqrt{324n^2 + 100^2}} = \frac{94984}{\sqrt{324n^2 + 10000}}.$$

10th, From (425),

From (436), a being zero, multiplying by 2,

Weight of bottom chords due 
$$\left.\right\} = \frac{5 \times 4 \times 100^3 \times 0.0225h}{18 \times 24 \times 18}$$

$$\left\{ \times \frac{2n^3 - 3n^2 + 22n - 24}{n^3} (n \text{ even}), \times \frac{2n^3 - 3n^2 + 22n - 21}{n^3} (n \text{ odd}). \right\}$$

11th, From (426), Q, being 8.181818 tons,

Weight of verticals, in pounds,

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Whn^{2} + \frac{5 \times 4 \times 200h}{2 \times 18 \times 8.1818} \left(\frac{7n^{2} + 3n - 10}{n}\right)$$

$$(n \text{ even}),$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^{2} - 1) + \frac{5 \times 4 \times 200h}{2 \times 18 \times 8.1818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$(n \text{ odd}).$$

Weight of verticals due wind =  $9000 \frac{h}{n^2}$ 

if d = 8 inches.

12th, From (428), 
$$\frac{1}{\sin^2 \phi}$$
 being equal to  $1 + \frac{l^2}{n^2 h^2}$ ,

Weight of girder diagonals
$$= \frac{4 \times 5 \times 200 \times 4h}{18 \times 24 \sin^2 \phi} \left(\frac{n^2 - 1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^2 \phi} W n^2$$

$$= \frac{4 \times 5 \times 200 \times 4h}{18 \times 24 \sin^2 \phi} \left(\frac{n^2 - 1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^2 \phi} W (n^2 - 1)$$

$$= \frac{(n \text{ odd})}{(n \text{ odd})}.$$

We therefore have, —

Weights in Pounds, W in Tons, A in Feet,  $\pi L = 200$  Tons.

	Top chords { Load Initial st.,	4140.742	$\frac{w}{h}$	_	Wh	165630	1 1	-	ħ°	-	
	Initial st.,	-	~	-		-	"	706		-	l
-	Bottom chords \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \	2444-444		-		97777	1	-	1	-	
		-				-		-	1	122,2222	1
1	Verticals	-		9.77778		-		-	1	777.1852	
ı	Girder diagonals	1333-333	ļ	3.33333		71111	ł	-		177.7778	
	K	-	_			-		63566		569.3262	_
	2000n W ==	7918.519		+13.11111		+334518		+64272		+1646.5114	
Ī	(Ined	4866.256		_		162208		_		-	=
l	Top chords { Load Initial st.,	<b>4000.230</b>		_		-	1	645	H	-	
l		2777.777		_		92592		_		-	
I	Bottom chords $\left\{ egin{array}{ll} { m Load} & { m Wind} & { m Const.} \end{array} \right.$	-/////		_		<b>9</b> -39-		_		115.7408	
Ì	Verticals	_		14.66667		-	İ	_		838.4774	
	Girder diagonals	1388.889		5.00000		60014		_		216.0494	
١	K	-		_		-		58635		518.4826	
ľ	2000nW =	9032.922	<del></del>	+19.66667		+314814		+59280		+1688.7502	-
j	m . (Load	5525.323		_		157866		_			=
	Top chords { Load Initial st.,			-		-5,722		590		_	
1		3174.604		_	[	90702	}	_		_	
	Bottom chords { Load . Wind .	_		-		-		_		113.3788	
	Verticals	-	}	19.55556		_	ĺ	_		795.6160	
1	Girder diagonals	1360.544		6.66666		51830		_		253.9682	
	K	-		_		_		55614		479.7376	
ŀ	2000nW =	10060.471		+26,22222		+300398	-	+56204		+1642.7006	-

Weights in Pounds, W in Tons, k in Feet,  $\pi L = 200$  Tons.

									_		=
<b>%</b>	Top chords Load Initial st.,	6221.067	W/h	-	Wh	155526	I h	- 542	ħ°	-	A
	Bottom chords \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \			_		86976		- 3 <del>4-</del>		~	
	Wind.	-		-		-		-		111.2196	
	Verticals	_		26.07408		-		-		924.8842	l
	Girder diagonals	1388.889		8.88889		45574		-		291.6666	ı
	K	-		-		-		53732		463.3331	
	2000 <i>n IV</i> =	11168.983		+34.96297		+290076		+54274		+1791.1035	
9	Top chords { Load Initial st.,	6881.576		49		152924		-		-	Ī
7	Initial st.,	-		-		-		499		-	ı
	Bottom chords { Load . Wind .	3978.051				88402		-		-	l
	Wind.	-	Ì	-		-				110.5014	ı
	Verticals	-		32.59260		-		-		915.8665	
	Girder diagonals	1371.742		11.11111		40644		-		329.2180	
	<i>K</i>	-						52579		449.8421	
	2000 n W =	12231.369		+43.70371		+281970		+53078		+1805.4280	
											Ī
10	Top chords { Load Initial st.,	7564.819	ŀ	-		151296		-		-	l
ļ	-	-		-		-		461		-	l
Í	Bottom chords { Load . Wind .	4388.889		-		87778		-		-	ı
		-		-		-	H			109.7222	l
	Verticals	-		40.74075		-		-	1 1	1067.7777	l
	Girder diagonals			13.88889		3666 <del>7</del>		-	1	366.666 <sub>7</sub>	
	K	-		-		_		51915		448.8212	
	2000n IV =	13342.597		+54.62964		+275741		+52376		+1992.9878	
11	Top chords   Load	8226.202		-		149568		-		-	Ī
`	Top chords \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \			<b>-</b> -	1	-	l	428		-	l
-	Bottom chords { Load . Wind .	4820.936		-		87654		. <b>-</b>		•	l
		-		-		-		-		109.5668	İ
	Verticals	-		48.88888	į	-		-		1071.0130	
ı	Girder diagonals	1377.410		16.66667	ł	33392		-		404.0404	1
	<i>K</i>					-		51595		447.8397	
	2000n W =	14424.548		+65.55555		+270614		+52023		+2032.4599	
2	Ton should Load	8903.037		_		148384				-	
	Top chords { Load Initial st.,	-		-		-		399		-	
		5246.912		-		87448		-		-	1
ŀ	Bottom chords { Load . Wind .			-		-		-		109.3106	
	Verticals	-		58.66667		-		-		1232.6646	
	Girder diagonals	1388.889		20.00000		3065 <b>0</b>		-	1	441.3580	
							. 1		1		
	K	-		-		-		51521		455.0907	

Multiplying each of these 8 equations by  $(h \div 20000)$ , we find the uniform panel weight, W, of bridge, in terms of h, thus:

$$n = 5, W = \frac{16.7259 + 3.2136h + 0.08232557h^2}{-0.395926 + 0.5h - 0.00065555h^2}$$

$$n = 6$$
,  $W = \frac{15.7407 + 2.964h + 0.08443751h^2}{-0.4516467 + 0.6h - 0.00098333h^2}$ 

$$n = 7, W = \frac{15.0199 + 2.8102h + 0.08213503h^2}{-0.5030235 + 0.7h - 0.001311111h^2}$$

$$n = 8$$
,  $W = \frac{14.5038 + 2.7137h + 0.08955518h^2}{-0.5584491 + 0.8h - 0.001748148h^2}$ 

$$n = 9$$
,  $W = \frac{14.0985 + 2.6539h + 0.0902714h^2}{-0.6115685 + 0.9h - 0.002185185h^2}$ 

$$n = 10$$
,  $W = \frac{13.78705 + 2.6188h + 0.09964939h^2}{-0.66712985 + h - 0.002731482h^2}$ 

$$n = 11, W = \frac{13.5307 + 2.60115h + 0.10162299h^2}{-0.7212274 + 1.1h - 0.00327777h^2}$$

$$n = 12$$
,  $W = \frac{13.3241 + 2.596h + 0.11192120h^2}{-0.7769419 + 1.2h - 0.00393333h^2}$ 

Differentiating, and putting  $\frac{dW}{dh}$  = 0, according to equation (470), we find, —

Height, h, answering to Minimum Value of nW.

Span l = 100 Feet, Uniform Live Load nL = 200 Tons.

Weight of bridge, nW   59.97030   57.05609   54.55331   54.3868   14\frac{3}{7}   12\frac{1}{7}   12\	Number of Panels, #.	5	6	7	8
Ratio of length to height	Weight of bridge, nW	59.97030	57.05609	54-55331	54.38682
Ratio of dead to live load	Ratio of length to height	6.48060	6.84180	6.98290	7.44520
Weight of bridge per lin. ft., lbs., 1199.00000 1141.00000 1091.00000 188.0000 Weight of wood, tons	Ratio of dead to live load	0.29985	0.28528	0.27277	0.27193
Weight of wood, tons       25.55650       22.32550       20.99450       18.4688         Weight of iron, tons       34.41380       34.73060       33.55880       35.9183         Cost of iron, at \$150       \$5162 07       \$5209 59       \$5033 82       \$5387 7         Cost of wood, at \$15       383 35       334 88       301 42       277 0         Cost of bridge       5545 42       5544 47       5335 24       5664 7         Excess over least       210 18       209 23       0       329 5         Cost per linear foot       55 46       55 45       53 35       56 6         Number of Panels, n.       9       10       11       12         Height in feet, h.       13.10601       12.33301       12.04572       11.4037         Weight of bridge, nW.       53.61297       54.43510       54.39900       55.6463         Panel length, feet, l.       n.       11½       10.00000       917       8½         Ratio of length to height       7.63010       8.10830       8.30170       8.7690         Slope of diagonals, p.       49° 42′ 33″       0.27180       0.27199       0.2782         Ratio of dead to total load       0.21140       0.21394       0.21384       0.2176		1	1	· ·	0.21379
Cost of iron, at \$150 \$5162 o7 \$5209 59 \$5033 82 \$5387 7 Cost of wood, at \$15	Weight of wood, tons	25.55650	22.32550	20.99450	18.46850
Cost of wood, at \$15		34.41300	34.73000	33.55000	35.91030
Cost of bridge					\$53 <sup>8</sup> 7 75
Cost per linear foot	Cost of bridge		B	1	5664 78
Number of Panels, n.  9 10 11 12  Height in feet, h	<b>T</b>	i		1	329 54 56 65
Height in feet, h		33 40	33 43	33 33	30 03
Weight of bridge, $nW$ 53.61297 54.43510 54.39900 75.6465 81 11\frac{1}{3} 10.00000 9\frac{1}{1} 8\frac{1}{3} 8.7690 810830 8.30170 82.0000 9\frac{1}{1} 8\frac{1}{3} 8.7690 810830 8.30170 82.0000 9\frac{1}{1} 8\frac{1}{3} 8.7690 81.0830 8.30170 92.0000 9\frac{1}{1} 8\frac{1}{3} 8.7690 92.0000 92.0000 9\frac{1}{1} 8\frac{1}{3} 8.7690 92.00000 92.0000 92.0000 92.0000 92.0000 92.0000 92.0000 92.0000 92.00	Number of Panels, n.	9	10	11	10
Panel length, feet, I ÷ n					
Ratio of length to height	=	13.10601	12.33301		11.40376
Ratio of dead to live load 0.26806	Weight of bridge, nW	13.10601 53.61297	12.33301 54.43510	12.04572 54.39900	11.40376 55.64659
Ratio of dead to total load	Weight of bridge, $nW$ Panel length, feet, $l \div n$ Ratio of length to height	13.10601 53.61297 111 7.63010	12.33301 54.43510 10.00000 8.10830	12.04572 54.39900 911 8.30170	11.40376 55.64659 81 8.76900
Weight of wood, tons	Weight of bridge, $nW$ Panel length, feet, $l \div n$ Ratio of length to height  Slope of diagonals, $\phi$	13.10601 53.61297 111 7.63010 49° 42′ 33″	12.33301 54.43510 10.00000 8.10830 50° 57′ 49″	12.04572 54.39900 911 8.30170 52° 57′ 30″	11.40376 55.64659 81 8.76900 53° 50′ 33″
Weight offeron, tons	Weight of bridge, nW  Panel length, feet, $l \div n$ Ratio of length to height  Slope of diagonals, $\phi$ Ratio of dead to live load	13.10601 53.61297 111 7.63010 49° 42′ 33″ 0.26806	12.33301 54.43510 10.00000 8.10830 50° 57′ 49″ 0.27218	12.04572 54.39900 911 8.30170 52° 57′ 30″ 0.27199	11.40376 55.64659 81/8 8.76900 53° 50′ 33″ 0.27823
Cost of wood, at \$15 258 54 244 07 232 47 222 96	Weight of bridge, nW.  Panel length, feet, $l \div n$ Ratio of length to height.  Slope of diagonals, $\phi$ Ratio of dead to live load.  Ratio of dead to total load  Weight of bridge per lin. ft., lbs.,	13.10601 53.61297 111 7.63010 49° 42′ 33″ 0.26806 0.21140	12.33301 54.43510 10.00000 8.10830 50° 57' 49" 0.27218 0.21394 1089.00000	12.04572 54.39900 911 8.30170 52° 57′ 30″ 0.27199 0.21384 1088.00000	11.40376 55.64659 81/8 8.76900 53° 50′ 33″ 0.27823 0.21767 1113.00000
	Weight of bridge, $nW$ .  Panel length, feet, $l \div n$ .  Ratio of length to height.  Slope of diagonals, $\phi$ .  Ratio of dead to live load.  Ratio of dead to total load  Weight of bridge per lin. ft., lbs.,  Weight of wood, tons.	13.10601 53.61297 111 7.63010 49° 42′ 33″ 0.26806 0.21140 1072.00000 17.23600	12.33301 54.43510 10.00000 8.10830 50° 57' 49" 0.27218 0.21394 1089.00000 16.27100	12.04572 54.39900 911 8.30170 52° 57′ 30″ 0.27199 0.21384 1088.00000 15.49800	11.40376 55.64659 81/8 8.76900 53° 50′ 33″ 0.27823 0.21767 1113.00000 14.86500
I Cook of building	Weight of bridge, nW.  Panel length, feet, l:n  Ratio of length to height.  Slope of diagonals, p  Ratio of dead to live load.  Ratio of dead to total load  Weight of bridge per lin. ft., lbs.,  Weight of wood, tons  Weight of iron, at \$150.	13.10601 53.61297 1113 7.63010 49° 42′ 33″ 0.26806 0.21140 1072.00000 17.23600 36.37700	12.33301 54.43510 10.00000 8.10830 50° 57' 49" 0.27218 0.21394 1089.00000 16.27100 38.16410	12.04572 54.39900 911 8.30170 52° 57′ 30″ 0.27199 0.21384 1088.0000 15.49800 38.90100	11.40376 55.64659 81/8 8.76900 53° 50′ 33″ 0.27823 0.21767 1113.00000 14.86500
	Weight of bridge, nW.  Panel length, feet, l:n  Ratio of length to height.  Slope of diagonals, p  Ratio of dead to live load.  Ratio of dead to total load  Weight of bridge per lin. ft., lbs.,  Weight of wood, tons  Weight of iron, at \$150  Cost of wood, at \$15	13.10601 53.61297 111 7.63010 49° 42′ 33″ 0.26806 0.21140 1072.00000 17.23600 36.37700 \$5456 55 258 54	12.33301 54.43510 10.00000 8.10830 50° 57' 49" 0.27218 0.21394 1089.00000 16.27100 38.16410 \$5724 62 244 07	12.04572 54.39900 911 8.30170 52° 57′ 30″ 0.27199 0.21384 1088.00000 15.49800 38.90100	11.40376 55.64659 81/3 8.76900 53° 50′ 33″ 0.27823 0.21767 1113.00000 14.86500 40.78160 \$6117 24 222 98
Cost per linear foot	Weight of bridge, nW.  Panel length, feet, l:n  Ratio of length to height.  Slope of diagonals, p  Ratio of dead to live load.  Ratio of dead to total load  Weight of bridge per lin. ft., lbs.,  Weight of wood, tons  Weight of iron, at \$150.	13.10601 53.61297 111 7.63010 49° 42′ 33″ 0.26806 0.21140 1072.00000 17.23600 36.37700 \$5456 55 258 54 5715 09	12.33301 54.43510 10.00000 8.10830 50° 57' 49" 0.27218 0.21394 1089.00000 16.27100 38.16410 \$5724 62 244 07 5968 69	12.04572 54.39900 911 8.30170 52° 57′ 30″ 0.27199 0.21384 1088.00000 15.49800 38.90100 \$5835 15 232 47 6067 62	11.40376 55.64659 81/3 8.76900 53° 50′ 33″ 0.27823 0.21767 1113.00000 14.86500 40.78160 \$6117 24 222 98 6340 22

re, for weight, the minimum minimorum is 53.61297 tons,  $\phi = 49^{\circ} 42' 33''$ ; while for cost, at the assumed prices, st is \$5,335.24, answering to n = 7, and  $\phi = 45^{\circ} 4' 13''$ . nparing these results with the corresponding ones in 141, we conclude:—

For a given span and number of panels, if we increase load, we should increase the height.

As the live load increases, the ratio of dead to both live al loads diminishes.

As another example, let the span l = 200 feet; uniform d nL = 200 tons, or 1 ton per linear foot; other data as :les 141 and 142. Compute for n = 8, 9, 10, 11, 12, 13,

The floor weighs

$$F = \frac{2.5}{12} \times 17.5 \times 200 \times 50 = 36458$$
 pounds.

By (432),

ess of a joist, 
$$b = \left\{ \frac{9 \times 10 \times 2 \times 200}{\pi^2 \times 17.5 \times 7000} (36458 + 400000) \right\}^{\frac{1}{2}}$$

$$= \frac{10.51042}{\pi^{0.4}} \text{ inches.}$$

om (433),

of joists, 
$$J = \frac{200 \times 17.5 \times 50}{144 \times 2} \left( \frac{10.51042^3}{n^{1.4}} \right) = \frac{705523}{n^{1.2}}$$
 pounds.

Depth of I floor beams, from (412),

$$d = 3.80122 \left\{ \left( \frac{705523}{n^{1.2}} + 436458 \right) \frac{18.5 \times 4}{50000n} \right\}^{\frac{1}{2}}$$
$$= 0.4331885 \left( \frac{J + 436458}{n} \right)^{\frac{1}{2}} \text{ inches.}$$

By (434),

Weight of I-beams,

$$P = 15.46068(n-1) \times \frac{5}{18} \times 18.5 \left(\frac{J+436458}{n} \times \frac{18.5}{5}\right)$$
$$= 1.031824(n-1) \left(\frac{J+436458}{n}\right)^{\frac{5}{1}} \text{ pounds.}$$

4th, Top horizontal diagonals, as in article 142, weig

$$6.718\sqrt{324\pi^3 + 40000}$$
 pounds,

I now being 200.

Weight of bottom horizontal diagonals
$$= X = \frac{2 \times 5 \times 4 \times 18 \times 6}{18 \times 24 \sin^2 \phi_1} W_1 \begin{cases} n^3 & (n \text{ even}) \\ (n^3 - 1) \end{cases}$$

$$W_1 = \frac{\hbar w l}{2n}, \quad \frac{1}{\sin^2 \phi_1} = 1 + \frac{40000}{18^2 n^2}.$$

5th, Top horizontal struts, as before, with change of 100 to 200.

Cross-section of one, due initial strain, = 
$$S_1 = 10.19495 \sqrt{\frac{1}{1}}$$

From (461),

Cross-section of one, due wind, =  $2ac = 0.61 \frac{h^2}{n}$  square in

From (465),

Weight of n top horizontal struts

$$= 37.02857h^{2} + 611.697n^{2}\sqrt{\frac{1}{n^{2} + 123.4568}}$$
 pounds  
=  $7405.714\frac{h}{n} + 611.697n^{2}\sqrt{\frac{1}{n^{2} + 123.4568}}$  pounds,

by reason of (471).

6th, Weight to be added to floor beams, due to wind, is equal to

$$4P' = 15.416666h \left(1 + \frac{65.712}{d^2}\right) \begin{cases} n \ (n \text{ even}), \\ \frac{n^2 - 1}{n} \ (n \text{ odd}). \end{cases}$$

7th, Weight to be added to posts to resist wind is equal to

$$\frac{12^2 \times 5 \times 4 \times 0.0225 \times 200h^3}{18 \times 25 \times 8 \times 2} = 1.8h^3 = \frac{72000}{n^2}h \text{ pounds.}$$

by reason of (471).

8th, Braces. From (459), if b = 5 feet,  $d_2 = 4$  inches.

Weight of braces

$$= \frac{4 \times 6 \times 5 \times 0.0225 \times 200h^{2}}{18 \times 18 \times 0.70711^{2}} \left(1 + \frac{60^{2}}{3000 \times 4^{2}}\right) = 3.58333h^{2}$$

$$= \frac{716.666}{n}h,$$

by (471).

# Computing for 8 values of n, we find, —

Weights of Components of K, in Pounds. l = 200 Feet,  $\pi L = 200$  Tons.

n	8	9	10	11
Floor Joists  I floor beams Do. from wind  Horizontal top struts  Horizontal diagonals  Braces Residual  K	36458.0000 58184.0000 11294.0000 150.9538½ 2859.0000 925.7142½ 1656.0000 527.2219½ 89.5833½ 2000.0000 1693.4732½ +112451	36458.0000 50515.0000 11809.0000 170.5809½ 3465.0000 822.8571½ 1729.0000 504.8319½ 79.6296½ 2000.0000 1577.8995½ +105976	36458.0000 44516.0000 12282.0000 194.9868½ 4092.0000 740.5714½ 1808.0000 502.7783½ 71.6666½ 2000.0000 1510.0031½ +101156	36458.0000 39705.0000 12721.0000 215.9620/2 4734.0000 673.2467/2 1891.0000 495.8917/2 65.1515/2 2000.0000 1450.2519/2 +97509
я	12	18	14	15
Floor	36458.0000 35768.0000 13131.0000 240.9957h 5386.0000 617.1428h 1978.0000 501.4819h 59.7222h 2000.0000 1419.3426h +94721	36458.0000 32492.0000 13518.0000 263.1340/2 6031.0000 569.6703/2 2067.0000 503.6705/2 2000.0000 1391.6030/2 +92566	36458.0000 29727.0000 13884.0000 288.8560h 6708.0000 528.9796h 2161.0000 512.2314h 51.1905h 2000.0000 1381.2575h +90938	36458.0000 27365.0000 14231.0000 312.0545/k 7373.0000 493.7143/k 2257.0000 520.3630/k 47.7777/k 2000.0000 1373.9095/k +89684

9th, Taking Q=16.7442 tons, as before, and  $L=\frac{l}{n}=\frac{200}{n}$  tons, we find

ht of top chords due vertical 
$$= \frac{5 \times 4 \times 200^{3}}{2 \times 18 \times 16.7442h} \left(W + \frac{200}{n}\right)$$
$$\times \frac{2n^{3} + 3n - 2}{n} (n \text{ even}),$$
$$\times \frac{2n^{3} + 3n^{2} - 2n - 3}{n^{2}} (n \text{ odd}).$$

train throughout each top chord due to initial strain of 0.99402 = 5.96412 tons, along each diagonal between top is, is

5.96412 cos \( \phi\_1 \) tons.

llowed pressure on top chords = 
$$\frac{16.7442}{4}$$
  
= 4.18605 tons per square inch.

ional cross-section of iron for oth top chords due to initial rain on top diagonals
$$= \frac{2 \times 5.96412}{4.18605} \cos \phi_{1}$$

$$= \frac{569.904}{\sqrt{324n^{2} + 40000}} \text{ square inches.}$$

ional weight for p chords due inial strain on top lagonals, pounds 
$$= \frac{12 \times 200 \times 5 \times 569.904}{18\sqrt{324n^3 + 200^2}} = \frac{379936}{\sqrt{324n^2 + 40000}}$$

oth, From (425),

ht of bottom chords due ertical forces, pounds
$$= \frac{5 \times 4 \times 200^{2}}{2 \times 18 \times 24 h} \left( W + \frac{200}{n} \right)$$

$$\times \frac{2n^{3} - 3n^{2} + 22n - 24}{n^{2}} (n \text{ even}),$$

$$\times \frac{2n^{3} - 3n^{2} + 22n - 21}{n^{3}} (n \text{ odd}).$$

From (436), a being zero, multiplying by 2,

Weight of bottom chords due wind, in pounds 
$$= \frac{5 \times 4 \times 200^{3} \times 0.0225h}{18 \times 24 \times 18}$$
 
$$\times \frac{2n^{3} - 3n^{2} + 22n - 24}{n^{3}} (n \text{ even}),$$
 
$$\times \frac{2n^{3} - 3n^{2} + 22n - 21}{n^{3}} (n \text{ odd}).$$

11th, From (426), Q, being 8.181818 tons,

Weight of verticals due load, pounds,

$$= \frac{3 \times 5 \times 4Whn^{2}}{18 \times 8.181818} + \frac{5 \times 4 \times 200h}{2 \times 18 \times 8.181818} \left(\frac{7n^{2} + 3n - 10}{n}\right)$$

$$= \frac{3 \times 5 \times 4Wh(n^{2} - 1)}{18 \times 8.181818} + \frac{5 \times 4 \times 200h}{2 \times 18 \times 8.181818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4Wh(n^{2} - 1)}{18 \times 8.181818} + \frac{5 \times 4 \times 200h}{2 \times 18 \times 8.181818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4Wh(n^{2} - 1)}{18 \times 8.181818} + \frac{5 \times 4 \times 200h}{2 \times 18 \times 8.181818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

$$= \frac{3 \times 5 \times 4Wh(n^{2} - 1)}{18 \times 8.181818} + \frac{5 \times 4 \times 200h}{2 \times 18 \times 8.181818} \left(\frac{7n^{3} - 3n^{2} - 7n + 3}{n^{2}}\right)$$

Weight of verticals due wind, pounds, by (467),

$$= \frac{144 \times 5 \times 4 \times 0.0225 \times 200}{18 \times 2 \times 25 \times 8} h^3 = \frac{72000h}{n^2}$$

approximately, (471).

12th, From (428), where 
$$\frac{1}{\sin^2 \phi} = 1 + \frac{l^2}{n^2 h^2}$$
,

Weight of girder diagonals

$$= \frac{4 \times 5 \times 200 \times 4h}{18 \times 24 \sin^2 \phi} \left(\frac{n^2 - 1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^2 \phi} W n^2$$

$$(n \text{ even}),$$

$$= \frac{4 \times 5 \times 200 \times 4h}{18 \times 24 \sin^2 \phi} \left(\frac{n^2 - 1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^2 \phi} W (n^2 - 1)$$

$$(n \text{ odd}).$$

#### MECHANICS OF THE GIRDER.

Weights in Pounds, W in Tons, A in Feet,  $\pi L = 800$  Tons.

										_
4.Tand	24884	W		WA	622100	T	_	4.		Ī
horda Lond .	24004	W/A	_	T A	023100	Ä	1542	•	_	ľ
	14236	1	_		254000	П	1342	Ш	_	l
n chords   Load .   Wind .	14230	ı	_		353900	١,		Ш	889.757	ı
nis	_	1	26.0741		_		_		1909.258	l
diagonals	5556		8.8889		180990		_	Ш	291.667	ľ
K	2224	1	0.000g	1	1000,0		119451		1693.473	ŀ
		⊢		<u> </u>		_				ŀ
2000# W ==	44676		+34 9630		+1100333	ı	+113993		+4784-155	ſ
		4	<u> </u>	1	<del></del>	:	1	<del></del>	<u></u>	÷
hords Load lnitial st.,	27526	1	_	l	611689	ì	-		~	۱
loitial st.,	-	ш	-	ŀ	- 1	ŀ	1476	]	-	l
n chords { Load . Wind .	15912	ш	-		353600	ŀ	-	١.	_	۱
Wind.	-	ш	-		-		-		884.ota	L
als	-	ш	32,5926		_		-		1693,644	ı
· diagonals	5487		11.1111		169577	ı	-		329.218	l
K	-		-		_	l	105976	ļ	1577-900	l
2000# W =	.0	1—				_	]			ŀ
2000/21/ == ]	48995		+43.7037		+1127866		+107452		+4484.774	l
		1	_ · _ ·	1	l l	ī	Ī	f	1	ī
hords { Load Initial st.,	30429	ı	-	ı	605180	İ	-		-	Ł
	-		-	l	-	l	1419		-	ĺ
n chords   Load . Wind .	17556	ı	-	l	351120	l	-		_	L
· I	-	l	-	Į.	-	ı	-		877.778	ŀ
als	-	[	40,7408	Į .	-	l	-		1697.778	ŀ
r diagonals	5556		13.8889	Į	146667	l	-		366.66 <del>7</del>	ļ
K	-		-	-	-	L	101156		1510,003	
2000# W =	533 <b>7</b> 1		+54.6297	,—	+1102967	1-	+102568	-	+4452.226	17
	3337-	<u> </u>	13411-31		,	<u>!</u>	11200	<u> </u>		Ī
		1	ĺ	l	Í	١	1	1	1	ì
hords Load Initial st.,	32905		-	E	598272	۱	-	1	_	l
		i i	-	ĺ	6-8	l	1350	ı	_	l
n chords   Load . Wind .	19284	ı	1 -		350618		-	Н		l
	-	ı	.0.000		-	Н	-		876.534	t
≄ls	_	ı	48,8889			i i	-		1592.675	L
r diagonals	5510		16.6666		133567	ŀ		П	404.040	l
K					-		97509	l	1450,252	L
114	57699		+63.5555		+1082457		+98859		+4322.501	
2000# IV =	31-32					_		_		Ŧ
		Ī	-	-	503533		_		_	İ
	356ta		-		593533		1201		-	
hords Load Initial st.,	35612		-		<u> </u>		1991		-	
hords Load Initial st.,			-		593533 349800		1291		- - - 874.486	
hords Load Initial st.,	35612 - 20988 -				<u> </u>		1991		- - 874.486 2670.165	
hords   Load   Initial st., a chords   Load .   Wind .	35622 20988 -		58.6667		349800		1291		1670,165	
hords Load	35612 - 20988 -		- - - - 58.6667 20,0000		<u> </u>		-		1670,165 441-357	
hords   Load   Initial st., a chords   Load .   Wind .	35622 20988 -				349800		1291   94721 +-96022		1670,165	-

					-						
# 13	Top chords { Load Initial st.,	38260 -	$\frac{W}{h}$	-	Wh	5886x5 -	I A	- 1235	ˡ	- -	h
	Bottom chords { Load . Wind .	22801		- -		350785		-	Н	9-4-000	ļ
	Verticals	_		68.4444		_				874.930 1614.025	
	Girder diagonals	5523		23.3333	1	113286		-		478.633	
	K.	-		-3.3333		-		92566		1 <b>391.6</b> 03	
	2000m IV =	66584	-	+91.7777		+1052686		+938oz		+4359.191	
	Top shords   Load	40952		-		585028		-		-	
14	Top chords Load Initial st.,	-	ł	-		-	ł	1181		_	
	Bottom chords { Load .   Wind .	24490	1	-	t l	349857		-	]	-	
	Wind.	-		-	•	-		-	1	<b>874.6</b> 36	
	Verticals	-		79.8519		-		-		<b>172</b> 9.182	
	Girder diagonals	5556		27.2222		105280		-	İΙ	515.874	
	<i>K</i>	-	_					90938		1381.257	
	2000n W =.	70998		+107.0741		+1040165		+92119		+4500.949	
15	Top chords { Load Initial st.,	43602		-		581360		-		-	
-3	Initial st.,	-		-	1   - -	-	ŀ	1131		-	
	Bottom chords { Load .   Wind .	26272		-		350293		-		-	
	•	-		-	ļ	-	1	-		875.720	
	Verticals	-		91.2593		-	1	-		1699.028	
	Girder diagonals	553 <sup>1</sup>	1	31.111	]	98326		-		<b>553.08</b> 6	
	<i>K</i>	<b></b>		-		-		89684		1373.909	
	2000# W =	75405		+122.3704		+1029979		+90815		+4501.743	

Weights in Pounds, W in tons, k in Feet, nL = 200 Tons.

Multiplying each of these 8 equations by  $(h \div 20000)$ , we find the uniform panel weight, W, of bridge, in terms of h, thus:

$$n = 8$$
,  $W = \frac{58.0146 + 5.69965h + 0.2392078h^2}{-2.2338 + 0.8h - 0.00174815h^2}$ 

$$n = 9, W = \frac{56.3933 + 5.3726h + 0.2242387h^2}{-2.44625 + 0.9h - 0.002185185h^2}$$

$$n = 10, W = \frac{55.14835 + 5.1284h + 0.2226113h^2}{-2.66855 + h - 0.002731485h^2}$$

$$n = 11, W = \frac{54.12285 + 4.94295h + 0.216125h^2}{-2.88495 + 1.1h - 0.00327777h^2}.$$

$$n = 12, W = \frac{53.2966 + 4.8006h + 0.2202675h^2}{-3.1078 + 1.2h - 0.00393333h^2}.$$

$$n = 13, W = \frac{52.6343 + 4.69005h + 0.2179595h^2}{-3.3292 + 1.3h - 0.00458888h^2}.$$

$$n = 14, W = \frac{52.00825 + 4.60595h + 0.2250475h^2}{-3.5499 + 1.4h - 0.005353705h^2}.$$

$$n = 15, W = \frac{51.49895 + 4.54075h + 0.2250872h^2}{-3.77025 + 1.5h - 0.00611852h^2}.$$

Differentiating these equations according to the form (470), and solving for h and W, we find as follows:—

HEIGHT, h, ANSWERING TO MINIMUM VALUE OF nW. Span l = 200 Feet, Uniform Live Load nL = 200 Tons.

Number of Panels, #.	8	9	10	11
Height in feet, $h$	163.83300 25.00000 10.29700 37° 50′ 46″ 0.81900 0.45030	155.38700 223 10.30800 41° 7′ 33″ 0.77700 0.43720	151.80500 20.00000 10.50800 43° 34′ 48″ 0.75900 0.43150	147.73400 18 <del>11</del> 10.58500 46° 5′ 54″ 0.73800 0.42490
Weight of wood, tons	47.32100 116.51200 \$17476 80 709 82	43.48700 111.90000 \$16785 00 652 31	40.48700 111.31800 \$16697 70 607 31	38.08100 109.65300 \$16447 95
Excess over least	1167 45	418 14	285 84	0

HEIGHT, h, ANSWERING TO MINIMUM VALUE OF nW. — Concluded.

Span l = 200 Feet, Uniform Live Load nL = 200 Tons.

Number of Panels, **.	12	18	14	15
_	147.06400 16 <del>}</del> 10.84100 47° 54′ 24″	15 <del>/3</del> 10.94900 49 <sup>0</sup> 53' 42"	146.09100 14 <sup>3</sup> 11.23300 51 <sup>0</sup> 15' 29"	11.36600 52° 50′ 52″
Ratio of dead to live load Ratio of dead to total load Weight of bridge, lbs. to lin. ft., Weight of wood, tons Weight of iron, tons	0.42370 1471.00000 36.11300	0.42070 1453.00000 34.47500	0.42210 1461.00000 33.09200	1455.00000
Cost of iron, at \$150 Cost of wood, at \$15 Cost of bridge	541 71 17184 36 165 19		496 38 17446 23 427 06	478 68 17518 53 499 36

Of the bridge weights in this case, the minimum minimum minimorum is 145.260 tons, n = 13,  $\phi = 49^{\circ} 53' 42''$ ; while of the costs at the assumed prices, the least is \$17,019.17, corresponding to n = 11,  $\phi = 46^{\circ} 5' 54''$ .

144. From articles 141 and 143, exemplifying 2 bridges of different spans but under the same live load per linear foot, we may deduce,—

1st, That, as the length increases, the bridge weight per linear foot increases; or, the ratio of dead to live load increases nearly as the length.

2d, That the dead load increases nearly as the square of the length.

3d, That an odd number of panels is more favorable to weight than an even number.

4th, That the height of each panel should be a little greater than its length.

5th, That the ratio of length to height of girder depends upon the span, as well as upon the live load, seen by comparing articles 141, 142, 143.

These principles are to be seen in this table.

### COMPARATIVE VIEW OF RESULTS.

Span, Feet,	Uniform Live Load, Tons, #L.	Best Number of Panels,	Best Height of Girder, Feet, &.	Least Weight of Bridge, Tons, nW.	Slope of Diagonals,	i *	Bridge Weight per Lin. Ft., Pounds.	Length to	
100	100	9	11.592	37.835	46° 12′ 51″	0.378	. 757	8.626	11 <del>]</del>
100	200	9	13.106	53.613	49° 42′ 33″	0.268	1072	7.630	11 <del>5</del>
200	200	13	18.266	145.260	49° 53′ 42″	0.726	1453	10.949	15 <del>13</del>

These examples may suffice to illustrate a mode of determining economical proportions for girders of all classes.

#### SECTION 2.

The Pratt Truss of Single System under Varying Live Load, without taking Account of Wind Pressure.

145. We shall here resume the example of article 36, the span being 100 feet of 10 panels, and the live load 2 locomotives of given weight and wheel base.

Take n = number of panels.

W =unknown panel weight of bridge.

h = 20 feet = height of girders, pin to pin.

q = 14 feet = width of bridge, in clear.

 $q_1 = 16$  feet = width of bridge, extreme.

Single track, 2 rails, 56 pounds per yard each.

Ties,  $6 \times 8 \times 84$  inches, spaced 8 inches in clear.

2 track stringers,  $12 \times d$  inches each.

Ties and stringers, pine, 40 pounds per cubic foot.

Weight of 2 rails =  $2 \times 100 \times \frac{56}{8} = 3,733$  pounds.

Weight of 75 ties =  $\frac{6 \times 8}{144} \times 75 \times 7 \times 40 = 7,000$  pounds.

Panel length of stringers = 120 inches.

Panel weight of rails = 373 pounds.

Panel weight of ties = 700 pounds.

2  $\times$  weight on 1 pair of drivers = 42,000 pounds distributed.

Maximum weight on 2 stringers = 43,073 pounds uniformly distributed.

Then, for both stringers,

$$b = breadth = 24$$
 inches.

$$d = \text{height} = 15 \text{ inches.}$$

Take f = 10 = factor of safety for pine.

B = 8,000 =breaking-weight for pine.

From equation (52),

$$M = \frac{1}{8}wl^2 = \frac{1}{8} \times 43073 \times 120,$$

where wl = 43,073 pounds; and, from (160),

$$R + f = \frac{1}{60}Bbd^2 = \frac{24 \times 8000}{60}d^2,$$

$$\therefore d^2 = \frac{43073 \times 120 \times 60}{8 \times 24 \times 8000},$$

$$d = 14.21$$
 inches.

Call d = 15 inches,

Weight of 2 stringers = 
$$\frac{2 \times 12 \times 15 \times 100 \times 40}{144}$$
 = 10000 pounds.

Suppose 2 wrought-iron I-beams suspended at each panel joint, and assume the load on these beams to be concentrated at their centre.

atest load on 2 beams,

From rails and ties, 1073 pounds, From stringers, 1000 pounds,

From locomotive, 28612 pounds (article 36),

Total, 30685 pounds,

re has the momental effect of 61,370 pounds uniformly ited along the double beam.

ice, for each single I-beam, D of (412) is equal to 30,685, and  $q_1 = 16$  feet.

e f = 6 = factor of safety.

B = 50,000 pounds.

412),

 $80122 \left(\frac{30685 \times 16 \times 6}{50000}\right)^{\frac{1}{2}} = 14.79 \text{ inches} = \text{required depth.}$ 

cross-section of 1 beam =  $S = 1.28839 \left( \frac{30685 \times 16 \times 6}{50000} \right)^{\frac{3}{2}}$ = 19.508 inches.

v the "heavy 15-inch I-beam" of the Union Iron Mills, rgh, Penn., weighs 67 pounds to the foot, and its section tently  $= 67 \times \frac{8}{10} = 20.1$  inches.

will, therefore, use the heavy 15-inch beam of 67 pounds oot.

ght of 9 pairs 15-inch I-beams, 67 pounds, 16 feet;

 $2 \times 9 \times 16 \times 67 = 19296$  pounds.

of 11 head struts, 14 feet, 20 pounds;

11 × 14 × 20 = 3080 pounds.

of 40 horizontal diagonals, 11 diameter, 3.359 pounds,

40 × 18 × 3.359 = 2419 pounds.

Weight of the residue,

10  $\times$  200 = 2000 pounds.

## RECAPITULATION.

Rails	=	3733	pounds,
Ties	=	7000	pounds,
Stringers	=	10000	pounds,
Beams	==	19296	pounds,
Head struts	=	3080	pounds,
Horizontal diagonals	=	2419	pounds,
Residue	=	2000	pounds.
K	=	47528	pounds.

For determining the girder strains in this example, we have already found, article 36, the greatest moments and greatest differences of moment due the given rolling-load.

From (65), we have the moments due W,

$$M = \frac{Wl}{2n}r(n-r) = 5Wr(10-r),$$

```
M_{1} = 45W + 478.32, 	 H_{1} = 2.25W + 23.916 	ext{ tons}; 	 M_{2} = 80W + 863.71, 	 H_{2} = 4.00W + 43.186 	ext{ tons}; 	 M_{3} = 105W + 1105.88, 	 H_{3} = 5.25W + 55.294 	ext{ tons}; 	 M_{4} = 120W + 1236.95, 	 H_{4} = 6.00W + 61.848 	ext{ tons}; 	 M_{5} = 125W + 1276.38, 	 H_{5} = 6.25W + 63.819 	ext{ tons}. 	 \Delta_{1}H = 2.25W + 23.916 = 	ext{ difference of horizontal strains.} 	 \Delta_{2}H = 1.75W + 19.270 	 \Delta_{3}H = 1.25W + 14.050 	 \Delta_{4}H = 0.75W + 10.451 	 \Delta_{5}H = 0.25W + 7.395 	 \Delta_{6}H = 4.785 	 \Delta_{7}H = 2.579 	 \Delta_{8}H = 0.932 	 \Delta_{9}H = 0.058
```

et  $\phi$  = angle of elevation of any diagonal,

$$\therefore \tan \phi = \frac{20}{10} = 2, \qquad \log \tan \phi = 0.3010300, \\ \log \sin \phi = 9.9515452, \\ \log \cos \phi = 9.6505152.$$

top chord, take ratio of panel length to least diameter; then, by (400),

$$Q = \frac{18}{1 + \frac{12^2}{3000}} = \frac{P}{S}$$
 of the Gordon formula = 17.176 tons.

ake ratio of length of vertical to its least diameter = 40; by (400),

$$Q_1 = \frac{18}{1 + \frac{40^2}{3000}} = 11.74 \text{ tons.}$$

et T = 50,000 pounds = 25 tons = limit of tension. f = 5 = factor of safety.unming the strains on the equal panel lengths, we have

'eight of top chords = 
$$\frac{2f}{Q}(23.75W + 248.063) \times 10 \times \frac{10}{3}$$
  
=  $460.93W + 4815$  pounds.

alling the strain on end panels of the bottom chord the as the strain on the adjacent panel, we have strains in m chord,

$$H_1 = 2.25W + 23.916$$
  
 $H_2 = 2.25W + 23.916$   
 $H_3 = 4.00W + 43.186$   
 $H_4 = 5.25W + 55.294$   
 $H_5 = 6.00W + 61.848$   
 $\Sigma H = 19.75W + 208.160$ 

... Weight of bottom chords, 
$$\frac{2f}{T}(19.75W + 208.16) \times 10 \times \frac{10}{3}$$
  
=  $263.33W + 2776$  pounds.

Strain on a vertical  $= Z = \Delta H \tan \phi$ ,

by the formulæ for Class IX.;

$$Z_{1} = 4.50W + 47.832$$

$$Z_{2} = 3.50W + 38.540$$

$$Z_{3} = 2.50W + 28.100$$

$$Z_{4} = 1.50W + 20.902$$

$$Z_{5} = 0.50W + 14.790$$

$$Z_{6} = 9.570$$

$$Z_{7} = 12.50W + 159.734$$

Weight of verticals,  $\frac{2f}{Q_1}$  (12.5 W + 159.734) × 20 ×  $\frac{10}{8}$ 

$$= \frac{2 \times 5}{11.74} (12.5W + 159.734) \times \frac{200}{8} = 709.86W + 9071 \text{ pounds,}$$

where  $Z_6$  is used twice to provide resistance to lateral shocks.

Strain on a diagonal =  $Y = \Delta H + \cos \phi$ .

If we call the strain on each of the first 5 counters equal to that on the fifth, from live load alone, we have

$$Y_{1} = Y_{2} = Y_{3} = Y_{4} = Y_{5} = 4.785 + \cos \phi,$$

$$Y_{6} = (7.395 + 0.25W) + \cos \phi,$$

$$Y_{7} = (10.541 + 0.75W) + \cos \phi,$$

$$Y_{8} = (14.050 + 1.25W) + \cos \phi,$$

$$Y_{9} = (19.270 + 1.75W) + \cos \phi,$$

$$Y_{10} = (23.916 + 2.25W) + \cos \phi.$$

$$\Sigma Y = (99.007 + 6.25W) + \cos \phi.$$

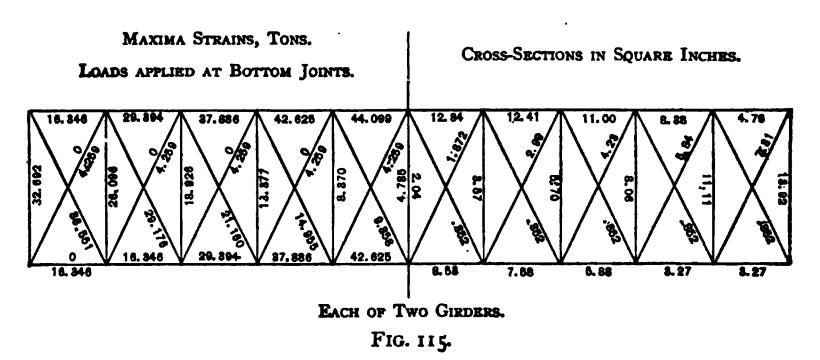
Weight of diagonals = 
$$\frac{2f}{T}(99.007 + 6.25 W) \times \frac{10 \times 10}{3 \cos^2 \phi}$$
  
= 416.17W + 6601 pounds.

#### RECAPITULATION.

Weight of top chords 
$$= 460.93W + 4815$$
 pounds, Weight of bottom chords  $= 263.33W + 2776$  pounds, Weight of verticals  $= 709.86W + 9071$  pounds, Weight of diagonals  $= 416.17W + 6601$  pounds, Weight of girders  $= G$   $= 1850.29W + 23263$  pounds,  $= 47528$  pounds, Weight of bridge  $= 2000nW = 1850.29W + 70791 = G + K$ .  $\therefore W = 3.9004$  tons  $=$  panel weight.

Weight of bridge = nW = 39.004 tons.

Substituting 3.9004 for W in the expressions for H, Y, and Z, we have this strain sheet:—



146. If the dead and live loads are applied at the upper joints, instead of the lower joints, the structure becomes a deck bridge; and the compressions here found for the verticals must be increased by the panel weight of dead load plus the greatest apex load from the locomotives; viz., for each girder we must augment Z by

$$\frac{3.9004 + 14.3063}{2} = 9.1034 \text{ tons.}$$

147. As another example of varying load applied at the lower joints of the Pratt Truss, we will, in accordance with the practice of some engineers, assume a certain panel weight of engine, of tender, and of train, and determine the strains thence resulting, and also the weight of the bridge.

Let us take the example given by Col. Merrill for this truss (see "Iron Truss Bridges for Railroads," by Col. William E. Merrill, U.S.A.); viz.,—

```
Span = 200 \text{ feet} = nc = l.
Length of panel = 12.5 \text{ feet} = c.
Height of truss = 18.75 \text{ feet} = h.
Number of panels = 16 = n.
On each of 2 trusses,
Panel weight of engine = 17,600 \text{ pounds}.
Panel weight of tender = 16,160 \text{ pounds}.
Panel weight of cars = 13,152 \text{ pounds}.
```

The engine is supposed to cover 2 panels, the tender 2 panels, and the cars follow. We therefore have,

```
1st panel weight of moving-load = W_5 = 8.800 tons, engine;
2d panel weight of moving-load = W_4 = 8.800 tons, engine;
3d panel weight of moving-load = W_3 = 8.080 tons, tender;
4th panel weight of moving-load = W_4 = 8.080 tons, tender;
5th panel weight of moving-load = W_4 = 6.576 tons, cars;
6th, etc., the same.
```

To find the strains due to this rolling-load, we employ equations (91); and for convenience, after dividing by the height, h = 18.75 feet, we may let  $r_1$  denote the number of panel weights of cars, and u = the number of panel weights of engine and tender on the girder at any time. We shall then have for the different positions of the load, by summing equations (91), and putting

$$X = (n - r_1 - 1) W_2 + (n - r_1 - 2) W_3 + (n - r_1 - 3) W_4 + \dots + (n - r_1 - 2) W_{n+1},$$

$$Y = \frac{r_1(r_1 + 1)}{2} W_1 + (r_1 + 1) W_2 + (r_1 + 2) W_3 + (r_1 + 3) W_4 + \dots + (r_1 + 3) W_4 + \dots + (r_1 + 2) W_{n+1}.$$

HORIZONTAL STRAINS AT JOINTS FOR ROLLING-LOAD.

$$H_{i} = \frac{c}{nh} \left\{ \left[ r_{i}n - \frac{r_{i}(r_{i}+1)}{2} \right] W_{i} + X \right\},$$

$$H_{2} = \frac{2c}{nh} \left\{ \left[ \left( r_{i} - \frac{1}{2} \right) n - \frac{r_{i}(r_{i}+1)}{2} \right] W_{i} + X \right\} = 2H_{i} - \frac{cW_{i}}{h},$$

$$H_{3} = \frac{3c}{nh} \left\{ \left[ \left( r_{i} - \frac{2}{2} \right) n - \frac{r_{i}(r_{i}+1)}{2} \right] W_{i} + X \right\} = 3H_{i} - \frac{3cW_{i}}{h},$$

$$H_{4} = \frac{4c}{nh} \left\{ \left[ \left( r_{i} - \frac{3}{2} \right) n - \frac{r_{i}(r_{i}+1)}{2} \right] W_{i} + X \right\} = 4H_{i} - \frac{6cW_{i}}{h},$$

$$H_{5} = \frac{5c}{nh} \left\{ \left[ \left( r_{i} - \frac{4}{2} \right) n - \frac{r_{i}(r_{i}+1)}{2} \right] W_{i} + X \right\} = 5H_{i} - \frac{10cW_{i}}{h},$$

$$H_{r_{1}} = \frac{r_{i}c}{nh} \left\{ \left[ \left( r_{i} - \frac{r_{i}-1}{2} \right) n - \frac{r_{i}(r_{i}+1)}{2} \right] W_{i} + X \right\}$$

$$= r_{i}H_{i} - \frac{(r_{i}-1)r_{i}}{2} \times \frac{cW_{i}}{h},$$

$$H_{r_{i}+1} = \frac{(r_{i}+1)c}{nh} \left\{ \left[ \left( r_{i} - \frac{r_{i}}{2} \right) n - \frac{r_{i}(r_{i}+1)}{2} \right] W_{i} + X \right\}$$

$$= (r_{i}+1)H_{i} - \frac{r_{i}(r_{i}+1)}{2} \cdot \times \frac{cW_{i}}{h},$$

$$H_{r_{i}+2} = \frac{c}{nh} \left\{ \frac{r_{i}(r_{i}+1)}{2} (n - r_{i}-2) W_{i} + (r_{i}+1)(n - r_{i}-2) W_{i} + (r_{i}+2)(n - r_{i}-2) W_{i} + (r_{i}+2)(n - r_{i}-2) W_{i} + (r_{i}+2)(n - r_{i}-2) W_{i} + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-2) W_{i} + \dots + (r_{i}+2)(n - r_{i}-2) W_{i} + \dots + (r_{i}+2)(n - r_{i}-2) W_{i} + \dots + (r_{i}+2)(n - r_{i}-2) W_{i} + \dots + (r_{i}+2)(n - r_{i}-2) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-2) W_{i} + \dots + (r_{i}+2)(n - r_{i}-2) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-2) W_{i} + \dots + (r_{i}+2)(n - r_{i}-2) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+2)(n - r_{i}-3) W_{i} + \dots + (r_{i}+3)(n - r_{i}-3) W_{i} + \dots + (r_{i}+3)(n - r_{i}-3) W_{i} + \dots + (r_{i}+3)(n - r_{i}-3) W_{i} + \dots + (r_{i}+3)(n - r_{i}-3) W_{i} + \dots + (r_{i}+3)(n - r_{i}-3) W_{i} + \dots + (r_{i}+3)(n - r_{i}-3) W_{i} + \dots + (r_{i}+3)(n - r_{i}-3) W_{i} + \dots + (r_{i}+3)(n - r_{i}-3) W_{i} + \dots + (r_{i}+3)(n - r$$

$$H_{r_1+3} = \frac{c}{nh} \left\{ \frac{r_1(r_1+1)}{2} (n-r_1-3) W_1 + (r_1+1) (n-r_1-3) W_2 + (r_1+2) (n-r_1-3) W_3 + (r_1+3) (n-r_1-3) W_4 + \dots + (r_1+3) (n-r_1-u) W_{u+1} \right\},$$

$$H_{r_1+4} = \frac{c}{nh} \left\{ \frac{r_1(r_1+1)}{2} (n-r_1-4) W_1 + (r_1+1) (n-r_1-4) W_2 + (r_1+2) (n-r_1-4) W_3 + (r_1+3) (n-r_1-4) W_4 + \dots + (r_1+4) (n-r_1-u) W_{u+1} \right\},$$

$$H_{r_1+u} = \frac{c}{nh} \left\{ \frac{r_1(r_1+1)}{2} (n-r_1-u) W_1 + (r_1+1) (n-r_1-u) W_2 + (r_1+2) (n-r_1-u) W_3 + (r_1+3) (n-r_1-u) W_4 + \dots + (r_1+u) (n-r_1-u) W_{u+1} \right\}$$

$$= \frac{c(n-r_1-u)}{nh} Y,$$

$$H_{r_1+u+1} = \frac{c(n-r_1-u-1)}{nh} Y,$$

$$H_{r_1+u+2} = \frac{c(n-r_1-u-2)}{nh} Y,$$

$$H_{n-1} = \frac{c}{nh} Y.$$

Computing these values of H from the above data, for every position of the train as it advances, a panel at a time, from left to right, preceded by the engine and tender, we find this table of horizontal strains at the joints, in tons; viz., as also given directly by the set of equations (91),—

#### MECHANICS OF THE GIRDER.

Hus	0.3	1.1	21.0	3-57₿	\$-257	7.313	9-440	11.94}	14.72	17.770	1500-16	\$69°\$z	±8.8€	32.71	37.134
Ни						:								65.42	68.4014
Hu	:					_							85.7	92.27	93.Bod
Нія												98.776	\$ go1	113.05	113.B16
$H_{11}$											105.47§	£17.60	125.3\$	128.84	198.44
$H_{10}$										106.624	130-7051	130.564	136.68	\$50.6E1	139.681
$H_{i}$									103-04	118.528	130.067	138.138	143-74	144.876	
H,							·	95-54	\$68-111	124.565	132-631	140-395	244-416	146.314	146.081] 144.546
H1		-					84-916	101.62	114.88	33.016	132-631	138.1283	141.708	143-369}	143.112
He						72.12	<b>88</b> ,54	Iot.8	172.48	rac.48	126.836	131.548	134.616	136.04	135.82
He					57-830\$	73-465	86-247	\$9.96	169-for	96-111	129-911	130.58	193-14	124.32	fb1-ber
$H_b$				42.92	\$7-221	68.944	78.568	B6.09	90.522	97-856	100.201	105-34\$ 130	107.28	108.229	rob.oht 124
H,			28,2E	fo.63	50.742	36-036	65.503	71.146	996-52	996-64	83-146	85.500	87.036	87.784	87.638
H <sub>3</sub>		18.4	24.57\$	32.474	38.88z)	43-741	48.052	31.814 <b>[</b>	55.0mg	969:45	\$9.8:4	61.385	60-40B	62.882	62.8opi
H <sub>1</sub>	5-5	10.6}	14-95	r8.93	ar.63\$	a4.062	26,218	<del>\$</del> 660°9€	29.70	31.04	3¢.099}	32.584	33-396	33.6}	33-598
Joints Israded,		1-3	1-3	t -1	I~ 5	9-1	1-7	8 -1	6 -1	1-10	Ē	1-13	1-t3	1=14	1-13

The blanks in this table may be filled by continually adding to itself each number in the right-hand column. It follows, therefore, that this right-hand column expresses the negative differences of simultaneous horizontal strains at adjacent joints due to rolling-load. It is evident, that, in this case, these negative differences are numerically the greatest differences of horizontal strains at adjacent joints, and may therefore be employed to find the maxima vertical and diagonal strains due live load.

The table shows (as was to be expected from this load, but contrary to the assumption made by Col. Merrill) that the horizontal strains are not maxima throughout when the foremost end of the engine is at the last joint, but that the greater part of these strains reach their greatest values when the foremost end (that is, forward panel weight) of the engine is at the fourteenth joint.

Since we require only the greatest horizontal strains, we need not compute the whole table, but only enough of the higher values of H to be certain that we find the highest at each joint. In the present example, it will suffice to compute values of H for the positions of load when  $r_1 = 11$ ,  $r_1 = 10$ ,  $r_1 = 9$ , and these for only the last 8 joints, since the first 7 horizontal joint strains are smaller than the last 7 by reason of the unequal loading.

We have, then, the following brief solution: -

1. For maxima differences of horizontal strain due live load.

$$H_{n-1} = H_{15} = \frac{c}{nh} Y = \frac{c}{nh} \left\{ \frac{r_1(r_1+1)}{2} W_1 + (r_1+1) W_2 + (r_1+2) W_3 + (r_1+3) W_4 + (r_1+4) W_5 \right\}.$$

c = 12.5,  $\lambda = 18.75$ , n = 16,  $W_1 = 6.576$ ,  $W_2 = W_3 = 8.08$ ,  $W_4 = W_5 = 8.8$ .

r <sub>1</sub>	<i>W</i> <sub>1</sub>	W <sub>2</sub>	W <sub>8</sub>	W4	W <sub>8</sub>	$H_{n-1}$	H <sub>15</sub>
-3	0	0	o	0	8.8	±4×8.8	0.3}
-2	0	0	0	8.8	8.8	$(\frac{1}{24} + \frac{9}{24})8.8$	1.1
<b>—</b> I	0	0	8.08	8.8	8.8	80.8× <del>1</del> +8.8× <del>1</del> +	2.17
0	0	8.08	8.08	8.8	8.8	<del>₹</del> 4×8.8+ <del>₹</del> 4×8.08	3-57
r	6.576	8.08	8.08	8.8	8.8	₽4×8.8+44×8.08+44×6.576	5-257
2	6.576	8.08	8.08	8.8	8.8	5.2573+14(2×8.08+2×8.8+2×6.576)	7.212
3	6.576	8.08	8.08	8.8	8.8	7.212 + 1/4 (33.76+3×6.576)	9.440 <b>}</b>
4	6.576	8.08	8.08	8.8	8.8	9.440g+1.40g+ <del>24</del> ×6.576	11.94
5	6.576	8.08	8.08	8.8	8.8	11.943+1.403+44×6.576	14.72
6	6.576	8.08	8.08	8.8	8.8	14.72 +1.40 <sup>2</sup> + 6×0.274	17.7703
7	6.576	8.08	80.8	8.8	8.8	17.7703+1.403+ 7×0.274	21.095
8	6.576	8.08	8.08	8.8	8.8	21.0953+1.403+ 8×0.274	24.694
9	6.576	8.08	8.08	8.8	8.8	24.694 +1.403+ 9×0.274	28.5
10	6.576	8.08	8.08	8.8	8.8	28.53+1.403+10×0.274	32.71
11	6.576	8.08	8.08	8.8	8.8	32.713+1.403+11×0.274	37-134

2. For the maxima horizontal strains due live load, as already computed and tabulated above.

```
(max)
(max)
                                                                      (max)
                                                                           (max)
(max)
                                                                                (max)
                                                                                       (max)
                                                                                                      \frac{1}{8}H_{13} =
                                                                                                                                                      11
                                                                                              Ş
                                                                                                              11
                                                                                        11
                                    H
                                                                                                       11
                                                                                                                                                       11
                                                                                                              ×××
                                                                          36 × 4.384
45 × 4.384
                                                                                                                                      36 \times 4.384
                       \times 4.384
                                                                                                                       0
                                                                                                                  0
                                                                                                                                                     W_1 + (n-r_1-1)W_2 + (n-r_1-2)W_3 + (n-r_1-3)W_4 + (n-r_1-4)W_5
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                                                                          = 139.05\
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             113.816
                  128.44
                       138.686
                                                                                                                                 136.68
                                                                                                                                      142.74
                                                                                                                                            144.416
                                                                                                                            125.233
                             144.546
                                                                                      146.314
                                                                                                                                                     33-396
        93.802
                                    146.021
                                                       65.42}
                                                            92.27$
                                                                                                                  85.7
    68.401
                                         33-59$
                                                  32.71$
                                                                                                             57.14
                                                                                                     28.53
                                                                                             33.6
```

It is manifest that the labor of computing the maxima values of H would be much lessened if we could legitimately assume that the horizontal strains are greatest throughout when the head of the engine is at the last joint or at any particular joint.

To find the strains due the unknown bridge weight, 2nW, we have the panel weight of bridge on each girder = W, and find from equation (65), after dividing by h,

$$H = \frac{W7}{2nh}r(n-r).$$

$$\therefore r = 0, H_0 = 0, \begin{cases} 1st \\ \text{difference.} \\ 5 \text{ W.} \end{cases}$$

$$r = 1, H_1 = 5 \text{ W.}$$

$$r = 2, H_2 = 9\frac{1}{3}W.$$

$$r = 3, H_3 = 13 \text{ W.}$$

$$r = 4, H_4 = 16 \text{ W.}$$

$$r = 5, H_5 = 18\frac{1}{3}W.$$

$$r = 6, H_6 = 20 \text{ W.}$$

$$r = 7, H_7 = 21 \text{ W.}$$

$$r = 8, H_8 = 21\frac{1}{3}W.$$

$$r = 8, H_8 = 21\frac{1}{3}W.$$

$$r = 181 \text{ W.}$$

$$r = 1, H_1 = 5 \text{ W.}$$

$$r = 2, H_2 = 9\frac{1}{3}W.$$

$$r = 3\frac{3}{3}\frac{3}{3}W.$$

$$r = 3\frac{3}{3}\frac{3}{3}W.$$

$$r = 4, H_4 = 16 \text{ W.}$$

$$r = 4, H_4 = 16 \text{ W.}$$

$$r = 5, H_5 = 18\frac{1}{3}W.$$

$$r = 6, H_6 = 20 \text{ W.}$$

$$r = 7, H_7 = 21 \text{ W.}$$

$$r = 8, H_8 = 21\frac{1}{3}W.$$

$$r = 8, H_8 = 21\frac{1}{3}W.$$

$$r = 18\frac{1}{3}W.$$

Panel.	Maximum Difference of Horizontal Strain = $\Delta H$ .	No.	Maximum Vertical Strain = $\Delta H \tan \phi$ = $\frac{3}{2}\Delta H$ .	Maximum Diagonal Strain.			
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16	0 $-5W$ 0.3\frac{3}{3} $-4\frac{1}{3}W$ 1.1 $-3\frac{2}{3}W$ 2.17 $-3W$ 3.57\frac{3}{3} $-2\frac{1}{3}W$ 5.257\frac{1}{3} $-1\frac{1}{3}W$ 7.212 $-W$ 9.440\frac{1}{3} $+\frac{1}{3}W$ 11.94\frac{1}{3} $+\frac{1}{3}W$ 11.94\frac{1}{3} $+\frac{1}{3}W$ 11.770\frac{1}{3} $+1\frac{1}{3}W$ 21.095\frac{1}{3} $+2\frac{1}{3}W$ 24.694 $+3W$ 28.5\frac{1}{3} $+3\frac{1}{3}W$ 32.71\frac{1}{3} $+4\frac{1}{3}W$ 37.134 $+5W$	9 10 11 12 13 14 15 16	$\frac{3}{2}(9.440\frac{3}{3} - \frac{1}{3}W)$ $\frac{3}{2}(11.94\frac{1}{3} + \frac{1}{3}W)$ $\frac{3}{2}(14.72 + W)$ $\frac{3}{2}(17.770\frac{3}{3} + 1\frac{3}{3}W)$ $\frac{3}{2}(21.095\frac{1}{3} + 2\frac{1}{3}W)$ $\frac{3}{2}(24.694 + 3W)$ $\frac{3}{2}(28.5\frac{3}{3} + 3\frac{3}{3}W)$ $\frac{3}{2}(32.71\frac{1}{3} + 4\frac{1}{3}W)$ $\frac{3}{2}(37.134 + 5W)$	$(11.94\frac{1}{3} + \frac{1}{3}W) \sec \phi$ $(14.72 + W) \sec \phi$ $(17.770\frac{3}{3} + 1\frac{3}{3}W) \sec \phi$ $(21.095\frac{1}{3} + 2\frac{1}{3}W) \sec \phi$ $(24.694 + 3W) \sec \phi$ $(28.5\frac{2}{3} + 3\frac{2}{3}W) \sec \phi$ $(32.71\frac{1}{3} + 4\frac{1}{3}W) \sec \phi$			

In the first 6 panels we shall introduce counters of 1 square inch cross-section, and therefore capable of resisting safely 5 tons where theoretically no strain appears. Also, we shall call the strain on the bottom chords in the first panel equal to that of the second panel; viz.,

$$5W + 37.134$$
 tons.

For each panel length of top chord, take ratio of length to least diameter = 15. Then the Gordon formula becomes, equation (400),

$$\frac{P}{S} = Q' = \frac{18}{1 + \frac{15^2}{3000}} = 16.744$$
 tons per square inch.

$$\frac{16.744}{5}$$
 = 3.3488 tons = allowed inch strain on top chords.

For each vertical strut, take ratio of length to least diameter = 37.5. Then

$$\frac{P}{S} = Q'' = \frac{18}{1 + \frac{37.5^2}{3000}} = 12.255$$
 tons per square inch.

$$\frac{12.255}{5}$$
 = 2.451 tons = allowed inch strain on verticals.

In tension, 5 tons allowed. Factor of safety f = 5 for all parts of girder; panel length of chords = 12.5 feet; length of verticals = 18.75 feet; length of diagonals = 23 feet; wroughtiron,  $\frac{5}{18}$  pound per cubic inch.

From these data we find,

Weight of top chords

$$= 4(124W + 872.244)$$

$$\times \frac{12.5 \times 10}{3.3488 \times 3} = 6171.36W + 43411$$
 pounds.

Weight of bottom chords

$$=4(107\frac{2}{8}W+763.06\frac{1}{8})$$

$$\times \frac{12.5 \times 10}{5 \times 3} = 3588.89W + 25435$$
 pounds.

Weight of verticals

$$= 4(21\frac{1}{8}W + 193.357\frac{2}{8}) \times \frac{2}{3}$$

$$\times \frac{18.75 \times 10}{2.451 \times 3} = 3238.47W + 29585 \text{ pounds.}$$

Weight of diagonals

$$= 4 \left( \frac{6 \times 5}{\sec \theta} + 20W + 205.290 \right)$$

$$\times \frac{23 \times 10 \sec \theta}{5 \times 3} = 2211.38W + 24539$$
 pounds.

Weight of 2 girders = 
$$G$$
 = 15210.10 $W$  + 122970 pounds.

W is in tons.

$$\tan \phi = \frac{18.75}{12.5} = \frac{3}{2}, \quad \sec \phi = \sqrt{1^2 + \left(\frac{3}{2}\right)^2} = 1.80278.$$

To find the constant part of the bridge weight = K. 2 rails, 200 feet, 56 pounds per yard, = 7,467 pounds. Rails 5 feet between centres.

Ties  $6 \times 8$  inches, 16 inches between centres.

 $\frac{12 \times 200}{16}$  = 150 ties, 7 feet long, 40 pounds per cubic foot.

Weight of ties = 150  $\times \frac{6 \times 8}{144} \times 7 \times 40 = 14,000$  pounds.

2 track stringers, each  $15 \times 18\frac{1}{2}$  inches, 40 pounds per cubic foot,  $\frac{2 \times 15 \times 18\frac{1}{2}}{144} \times 200 \times 40 = 30,833$  pounds.

Depth of stringer is thus found:—

Length between bearings = 12.5 feet.

Uniform load =  $\frac{1}{16}$  of rails and ties =  $\frac{21467}{16}$  = 1,342 pounds.

Concentrated load = panel weight of engine =  $2 \times 17,600$  = 35,200 pounds, which is equivalent to uniform load of 70,400 pounds.

... Uniform load on 2 stringers 30 inches wide = 71742 pounds.

For pine, take factor of safety = 10, and the ultimate inch resistance to cross-breaking B = 8,000 pounds;

... Moment due external forces =  $\frac{1}{8} \times 71742 \times 12 \times 12.5$ , from equation (52).

Moment of resistance due internal forces, equation (160),

$$= \frac{1}{6}Bbd^2 = \frac{8000 \times 30}{6}d^2,$$

which becomes, after introducing the factor of safety,

$$\frac{8000 \times 30d^2}{60} = 4000d^2.$$

Equating moments of external and allowable internal forces, we find

$$\frac{1}{8} \times 71742 \times 12 \times 12.5 = 4000d^2$$

 $\therefore \text{ Depth} = d = 18.338 \text{ inches, called } 18\frac{1}{4}.$ 

15 pairs I-beams, heavy 15-inch, 67 pounds, 16 feet, = 32,160 pounds.

Depth of beam is thus determined:

Panel weight of rails = 467 pounds,

Panel weight of ties = 875 pounds,

Panel weight of stringers = 1927 pounds,

Panel weight of engine = 35200 pounds.

Weight on each pair of I-beams = 38469 pounds.

Now, this weight is actually concentrated at two points 5 feet apart, under the rails, each point being  $5\frac{1}{2}$  feet from the nearer end of the I-beams.

The moment at the centre of the double beam is therefore, according to equation (43),

$$M_c = 2 \times 19234.5 \times \frac{8}{16} \times 5\frac{1}{2} \times 12 = \frac{1}{4}W' \times 16 \times 12$$

by equation (46), if W' is the weight at centre producing an equivalent moment;

$$W = 26447.4$$

and 52,895 pounds is the equivalent load uniformly distributed for 2 beams. Therefore uniform load on 1 beam is 26,447 pounds. And, if 6 is the factor of safety for beams, equation (412) gives, B being = 50,000,

Depth of beam = 
$$d_2 = 3.80122 \left( \frac{26447 \times 16 \times 6}{50000} \right)^{\frac{1}{2}} = 14.076$$
 inches.

From (413),

Area = 
$$1.28839 \left( \frac{26447 \times 16 \times 6}{50000} \right)^{\frac{9}{3}} = 17.667$$
 square inches.

Area of similar beam 15 inches deep = 
$$17.667 \times \frac{15^2}{14.076^2}$$

= 20.062 square inches.

Area of heavy 15-inch 67-pound beam of the Union Iron Mills, Pittsburgh, Penn.,

Hence we may with safety use this beam.

17 head struts, 14 feet, 25 pounds, = 5,950 pounds.

64 horizontal diagonals, 1\frac{1}{8} diameter, 19\frac{1}{2} feet, 3.359 pounds, = 4,192 pounds.

Residue, 200 pounds per panel, = 3,200 pounds.

### RECAPITULATION.

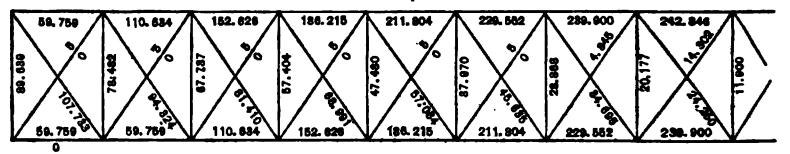
Weight of rails	=	7467 pounds,
Weight of ties	=	14000 pounds,
Weight of stringers	=	30833 pounds,
Weight of I-beams	==	32160 pounds,
Weight of head struts	=	5950 pounds,
Weight of horizontal diagonals	=	4192 pounds,
Weight of residue	=	3200 pounds.
K	=	97802 pounds.
Weight of girders	=	122970 + 15210.10W.
Weight of bridge	=	220772 + 15210.1 W pounds
	=	$2nW$ tons = $2 \times 16 \times 2000W$ pounds
		= 64000 W.

$$\therefore$$
 48789.9  $W = 220772$ ,  $W = 4.52493$  tons.

Weight of bridge = 32W = 144.798 tons.

Substituting this value of W in the expressions already found for greatest strains, we have,—

MAXIMA STRAINS, IN TONS, FOR EACH OF TWO GIRDERS.



LOADS APPLIED AT BOTTOM JOINTS.

Fig. 116.

To find the allowed cross-sections in square inches, we divide strains in top chord by 3.3488, strains in verticals by 2.451, and in bottom chord and diagonals by 5.

# CHAPTER X.

CALCULATION OF THE WEIGHT OF BRIDGES HAVING GIRDERS OF CLASS I., AND DETERMINATION OF THE NUMBER OF PANELS AND THE HEIGHT OF GIRDER, WHICH RENDER THE BRIDGE WEIGHT LEAST FOR A GIVEN SPAN AND UNIFORM LIVE LOAD. — LIMITING SPAN FOUND.

## SECTION I.

General Specifications for Iron Bridges, issued in 1879 by the New York, Lake Erie, and Western Railroad Company. O. Chanute, Chief Engineer.

148. General Specifications for Iron Bridges.

NEW YORK, LAKE ERIE, AND WESTERN RAILROAD COMPANY.

## 1879.

#### GENERAL DESCRIPTION.

- 1. All parts of the superstructure shall be of wrought-iron, except bed plates and washers, which may be of cast-iron.
- 2. The following modes of construction shall preferably be em- Kinds of ployed:— bridges.

Spans up to 17 feet . . . Rolled beams.

Spans 17 to 40 feet . . . Riveted plate girders.

Spans 40 to 75 feet . . . Riveted lattice girders.

Spans over 75 feet . . . Pin-connected trusses.

In calculating strains, the length of span shall be understood to be the distance between centres of end pins for trusses, and between centres of bearing-plates for all beams and girders. Spacing of girders.

3. The girders shall be spaced (with reference to the axis of the bridge) as required by local circumstances, and directed by the chief engineer of the railroad company.\*

Head room.

4. In all through bridges, there shall be a clear head room of 20 feet above the base of the rails.

Floor.

5. The wooden floor will consist of transverse floor timbers extending the full width of the bridge, supporting the rails and guard beams. Their scantling will vary with circumstances. They will be furnished and put on by the railroad company.

Loads.

- 6. Bridges shall be proportioned to carry the following loads:—
  1st, The weight of iron in the structure.
  - 2d, A floor weighing 400 pounds per lineal foot of track, to consist of the rails, ties, and guard timbers only.

These two items taken together shall constitute the "dead load."

3d, A moving-load for each track, supposed to be moving in either direction, and consisting of two "consolidation" engines coupled, followed by a train weighing 2,240 pounds per running foot; this "live load" being concentrated upon points distributed as in the diagram on p. 415.

Stresses.

The maximum strains due to all positions of the above "live load," and of the "dead load," shall be taken to proportion all the parts of the structure.

Lateral stresses.

7. To provide for wind strains and vibrations, the top lateral bracing in deck bridges, and the bottom lateral bracing in through bridges, shall be proportioned to resist a lateral force of 450 pounds for each foot of the span; 300 pounds of this to be treated as a moving-load.

The bottom lateral bracing in deck bridges, and the top lateral bracing in through bridges, shall be proportioned to resist a lateral force of 150 pounds for each foot of the span.

Temperature.

8. Variations in temperature to the extent of 150 degrees shall be provided for.

<sup>\*</sup> Generally, in through bridges, the clear width between trusses shall be 15 feet for single track, and 28 feet for double track. In deck bridges, and for the floor system of all bridges, the spacing between the centres of trusses and girders shall generally be as follows:—

_	_	Doubl <b>e</b>	TRACK.
DESCRIPTION.	SINGLE TRACK.	2 Trusses.	3 Trusses.
Deck truss bridges  Deck plate girders  Floor stringers		16 feet or over. 16 feet or over. 10 feet or over.	to feet or over. to feet or over. 8 feet or over.

The centres of beams and plate girders shall be not less than 4 feet (on either side) from the centre of the broad gauge track.

The standard distance between centres of tracks is 13 feet.

25100	9. All parts shall be so designed that the strains coming upon them can be accurately calculated. 10. Strain sheets and a general plan showing the Plans a	nd
22000	dimensions of the parts and general details must strain accompany each proposal.	
<b>\$3</b> 000	of working drawings must be submitted for approval	
27000	by the chief engineer of the railroad company before the work is commenced.	
\$3000	12. Unless otherwise specified, the form of truss Form of may be selected by the builder; but, to secure unitruss.	Ē
14560	formity in appearance, it is desired that all "through" trusses shall be built with inclined end posts.	
34660	13. In comparing competitive plans, the relative cost of the wooden floors required will be taken into	
34000	consideration.  14. The following clauses are all intended to apply	
34700	to iron construction. Parties proposing to substitute steel for particular parts will be required to furnish evidence of its strength, elasticity, uniformity in production, and adaptability to the intended purpose.	
15100	duction, and adaptability to the intended purpose.  PROPORTION OF PARTS.	
<b>\$\$</b> 000	<ol> <li>All parts of the structures shall be so propor- Tensile tioned that the maximum strains produced shall in no strains.</li> </ol>	
22000	case cause a greater tension than the following:	
<b>220</b> 00	Pounds per Sq Inch,	
20000	On lateral bracing	
-24790	On solid rolled beams, used as cross floor beams and stringers	
24900	On bottom chords and main diagonals 10000	
	On counter rods and long verticals 8000	
14560	On bottom flange of riveted cross girders, net section 8000	
\$4000	On bottom flange of riveted longitudinal plate girders over 20 ft. long, net section, 8000	
	On bottom flange of riveted longitudinal plate girders under 20 ft. long, net section, 7000 On floor beam hangers, and other similar	
	members liable to sudden loading 6000	

Fig. 117.

Compressive strains.

2. Compression members shall be so proportioned that the maximum load shall in no case cause a greater strain than that determined by the following formulæ:—

$$P = 1 + \frac{L^2}{40000R^2}$$
 for square end compression members.

$$P = 1 + \frac{L^2}{30000R^2}$$
 for compression members with one pin and one square end.

$$P = \frac{1 + \frac{L^2}{20000R^2}}$$
 for compression members with pin bearings.

P = the allowed compression per square inch of cross-section.

L = the length of compression member, in inches.

R = the least radius of gyration of the section, in inches.

3. The lateral struts shall be proportioned by the above formulæ to resist the resultant due to an assumed initial strain of 10,000 pounds per square inch upon all the rods attaching to them, produced by adjusting the bridge.

4. In beams and girders, compression shall be limited, as follows:-

· · · · · · · · · · · · · · · · · · ·	Pounds per Square Inch.
In rolled beams used as cross floor beams and stringers In riveted plate girders used as cross floor beams, gross section,	10000 6000
In riveted longitudinal plate girders over 20 feet long, gross section	6000
section	5000

5. Members subjected to alternate strains of tension and compression shall be proportioned to resist each of them. The strains, however, shall be assumed to be increased by an amount equal to eight-tenths of the least strain.

Shearingstrains. 6. The rivets and bolts connecting all parts of the girders must be so spaced that the shearing-strain per square inch shall not exceed 6,000 pounds, nor the pressure upon the bearing-surface exceed 12,000 pounds

per square inch of the projected semi-intrados (diameter x thickness of piece) of the rivet or bolt hole.

7. Pins shall be so proportioned that the shearing-strain shall not Bendingexceed 7,500 pounds per square inch, nor the crushing-strain upon the strains. projected area of the semi-intrados (diameter x thickness of piece) of any member connected to the pin be greater than 12,000 pounds per square inch, nor the bending-strain exceed 15,000 pounds per square inch, when the centres of bearings of the strained members are taken as the points of application of the strains.

- 8. In case any member is subjected to a bending-strain from local loadings (such as distributed floors on deck bridges), in addition to the strain produced by its position as a member of the structure, it must be proportioned to resist the combined strains.
- 9. Plate girders shall be proportioned upon the supposition that the Plate girders. bending or chord strains are resisted entirely by the upper and lower flanges, and that the shearing or web strains are resisted entirely by the web plate.
- 10. The compression flanges of beams and girders shall be stayed. against transverse crippling when their length is more than thirty times their width.
- 11. The unsupported width of any plate subjected to compression shall never exceed thirty times its thickness.
- 12. In members subject to tensile strains, full allowance shall be made for reduction of section by rivet holes, screw threads, etc.
- 13. The iron in the web plates shall not have a shearing-strain greater than 4,000 pounds per square inch, and no web plate shall be less than ‡ inch in thickness.
- 14. No wrought-iron shall be used less than  $\frac{1}{16}$  inch thick, except in places where both sides are always accessible for cleaning and painting.

#### DETAILS OF CONSTRUCTION.

- 1. All the connections and details of the several parts of the structure shall be of such strength, that, upon testing, rupture shall occur in the body of the members rather than in any of their details or connections.
- 2. Preference will be had for such details as will be most accessible for inspection, cleaning, and painting.
- 3. The web of plate girders must be spliced at all joints by a plate on each side of the web. T-iron must not be used for splices.
- 4. When the least thickness of the web is less than one-eightieth of the depth of a girder, the web shall be stiffened at intervals not over twice the depth of the girder.
- 5. The pitch of rivets in all classes of work shall never exceed 6 inches, nor sixteen times the thinnest outside plate, nor be less than three diameters of the rivet.

- 6. The rivets used will generally be ? and ? inch diameter.
- 7. The distance between the edge of any piece and the centre of a rivet hole must never be less than 1½ inches, except for bars less than 2½ inches wide; when practicable, it shall be at least two diameters of rivets.
- 8. When plates more than 12 inches wide are used in the flanges of plate or lattice girders, an extra line of rivets, with a pitch of not over 9 inches, shall be driven along each edge, to draw the plates together, and prevent the entrance of water
- 9. In punching plate or other iron, the diameter of the dye shall in no case exceed the diameter of the punch by more than  $\frac{1}{16}$  of an inch.
- 10. All rivet holes must be so accurately punched, that, when the several parts forming one member are assembled together, a rivet inch less in diameter than the hole can be entered, hot, into any hole, without reaming or straining the iron by "drifts."
  - 11. The rivets, when driven, must completely fill the holes.
- 12. The rivet heads must be hemispherical, and of a uniform size for the same-sized rivets throughout the work. They must be full and neatly made, and be concentric to the rivet hole.
  - 13. Whenever possible, all rivets must be machine-driven.
- 14. The several pieces forming one built member must fit closely together, and, when riveted, shall be free from twists, bends, or open joints.
- 15. All joints in riveted work, whether in tension or compression members, must be fully spliced, as no reliance will be placed upon abutting joints. The ends, however, must be dressed straight and true, so that there shall be no open joints.

Lower chords and suspension bars.

- 16. The heads of eye-bars shall be so proportioned that the bar will break in the body instead of in the eye. The form of the head and the mode of manufacture shall be subject to the approval of the chief engineer of the railroad company.
- 17. The bars must be free from flaws, and of full thickness in the necks. They shall be perfectly straight before boring. The holes shall be in the centre of the head, and on the centre line of the bar.
- 18. The bars must be bored of exact lengths, and the pin hole  $\frac{1}{20}$  inch larger than the diameter of the pin.
  - 19. The lower chord shall be packed as narrow as possible.

20. The pins shall be turned straight and sme

- 20. The pins shall be turned straight and smooth, and shall fit the pin holes within  $\frac{1}{60}$  of an inch.
- 21. The diameter of the pin shall not be less than two-thirds the largest dimension of any tension member attached to it. Its effective length shall not be greater than the breadth of the foot of the post plus four times the diameter of the pin. The several members attaching to the pin shall be packed close together, and all vacant spaces between the chords and posts must be filled with wrought-iron filling-rings.

Pins.

- 22. All rods and hangers with screw ends shall be upset at the ends, Upset screw so that the diameter at the bottom of the threads shall be 16 inch larger ends. than any part of the body of the bar.
- 23. All threads must be of the United States standard, except at the ends of the pins.
- 24. Floor beam hangers shall be so placed that they can be readily Floor beam examined at all times. When fitted with screw ends, they shall be pro-hangers. vided with check nuts.
- 25. When bent loops are used, they must fit perfectly around the pin throughout its semi-circumference.
- 26. Compression members shall be of wrought-iron of approved Compression members. forms.
- 27. The pitch of rivets, for a length of two diameters at the ends, shall not be over four times the diameter of the rivets.
- 28. The open sides of all trough-shaped sections shall be stayed by diagonal lattice work at distances not exceeding the width of the member. The size of bars shall be duly proportioned to the width.
- 29. All pin holes shall be re-enforced by additional material, so as not to exceed the allowed pressure on the pins. These re-enforcing plates must contain enough rivets to transfer the proportion of pressure which comes upon them.
- 30. Pin holes shall be bored exactly perpendicular to a vertical plane passing through the centre line of each member, when placed in a position similar to that it is to occupy in the finished structure.
- 31. The ends of all square-ended members shall be planed smooth, Abutting and exactly square to the centre line of strain.

joints.

- 32. All members must be free from twists or bends. Portions exposed to view shall be neatly finished.
- 33. The sections of the top chord shall be connected at the abutting Splicing of ends by splices sufficient to hold them truly in position.

- 34. In no case shall any lateral or diagonal rod have a less area than Lateral bracing. ? of a square inch.
- 35. The attachment of the lateral system to the chords shall be thoroughly efficient. If connected to suspended floor beams, the latter shall be stayed against all motion.
- 36. All through bridges with top lateral bracing shall have wrought- Transverse iron portals of approved design at each end of the span, connected diagonal rigidly to the end posts.

bracing.

- 37. When the height of the trusses exceeds 25 feet, overhead diagonal bracing shall be attached to each post and to the top lateral struts.
- 38. Pony trusses and through-plate or lattice girders shall be stayed by knee braces or gusset plates attached to the top chords, at the ends, and at intermediate points not more than 10 feet apart, and attached below to the cross floor beams or to the transverse struts.

In all deck bridges, diagonal bracing shall be provided at each panel. In double-track bridges, this bracing shall be proportioned to resist the unequal loading of the trusses. The diagonal bracing at the ends shall be of the same equivalent strength as the end top lateral bracing.

Bed plates.

39. All bed plates must be of such dimensions that the greatest pressure upon the masonry shall not exceed 250 pounds per square inch.

Friction rollers.

- 40. All bridges over 50 feet span shall have at one end nests of turned friction rollers formed of wrought-iron, running between planed surfaces. The rollers shall not be less than 2 inches diameter, and shall be so proportioned that the pressure per lineal inch of rollers shall not exceed the product of the square root of the diameter of the roller, in inches, multiplied by 500 pounds  $(500\sqrt{d})$ .
- 41. Bridges less than 50 feet span will be secured at one end to the masonry, and the other end shall be free to move by sliding upon planed surfaces.

Camber.

42. All bridges will be given a camber by making the panel lengths of the top chord longer than those of the bottom chord in the proportion of  $\frac{1}{2}$  of an inch to every 10 feet.

#### QUALITY OF IRON.

1. All wrought-iron must be tough, fibrous, and uniform in character. It shall have a limit of elasticity of not less than 26,000 pounds per square inch.

Finished bars must be thoroughly welded during the rolling, and free from injurious seams, blisters, buckles, cinder spots, or imperfect edges.

- 2. For all tension members, the muck bars shall be rolled into flats, and again cut, piled, and rolled into finished sizes. They shall stand the following tests:—
- Tension tests.

3. Full-sized pieces of flat, round, or square iron, not over 4½ inches in sectional area, shall have an ultimate strength of 50,000 pounds per square inch, and stretch 12½ per cent in their whole length.

Bars of a larger sectional area than  $4\frac{1}{2}$  square inches, when tested in the usual way, will be allowed a reduction of 1,000 pounds per square inch for each additional square inch of section, down to a minimum of 46,000 pounds per square inch.

4. When tested in specimens of uniform sectional area of at least \( \frac{1}{2} \) square inch for a distance of 10 inches taken from tension members which have been rolled to a section not more than  $4\frac{1}{2}$  square inches, the iron shall show an ultimate strength of 52,000 pounds per square inch, and stretch 18 per cent in a distance of 8 inches.

Specimens taken from bars of a larger cross-section than 4½ inches will be allowed a reduction of 500 pounds for each additional square inch of section, down to a minimum of 50,000 pounds.

- 5. The same-sized specimen taken from angle and other shaped iron shall have an ultimate strength of 50,000 pounds per square inch, and elongate 15 per cent in 8 inches.
- 6. The same-sized specimen taken from *plate* iron shall have an ultimate strength of 48,000 pounds, and elongate 15 per cent in 8 inches.
- 7. All iron for tension members must bend cold, for about 90 degrees, Bendingto a curve whose diameter is not over twice the thickness of the piece, tests. without cracking. At least one sample in three must bend 180 degrees to this curve without cracking. When nicked on one side, and bent by a blow from a sledge, the fracture must be nearly all fibrous, showing but few crystalline specks.
- 8. Specimens from angle, plate, and shaped iron must stand bending cold through 90 degrees, and to a curve whose diameter is not over three times its thickness, without cracking.

When nicked or bent, its fracture must be mostly fibrous.

- 9. Rivets and pins shall be made from the best double-refined iron.
- 10. The cast-iron must be of the best quality of soft gray iron.

Cast-iron.

- 11. All facilities for inspection of iron and workmanship shall be Tests. furnished by the contractor. He shall furnish, without charge, such specimens (prepared) of the several kinds of iron to be used as may be required to determine their character.
- 12. Full-sized parts of the structure may be tested at the option of the chief engineer of the railroad company; but, if tested to destruction, such material shall be paid for at cost, less its scrap value, to the contractor, if it proves satisfactory. If it does not stand the specified tests, it will be considered rejected material, and be solely at the cost of the contractor.

#### WORKMANSHIP.

- 1. All workmanship shall be first-class in every particular.
- 2. Abutting joints in truss bridges shall be in exact contact throughout.
- 3. Bars which are to be placed side by side in the structure shall be bored at the same temperature, and of such equal length, that, upon being piled on each other, the pins shall pass through the holes at both ends without driving.
- 4. Whenever necessary for the protection of the thread, provision shall be made for the use of pilot nuts, in erection.

### PAINTING.

- 1. All work shall be painted at the shop with one good coat of selected iron-ore paint and pure linseed-oil.
- 2. In riveted work, all surfaces coming in contact shall be painted before being riveted together.

#### MECHANICS OF THE GIRDER.

Bed plates, the inside of closed sections, and all parts of the work which will not be accessible for painting after erection, shall have two coats of paint.

- 3. Pins, bored pin holes, and turned friction rollers shall be coated with white lead and tallow before being shipped from the shop.
- 4. After the structure is erected, the ironwork shall be thoroughly and evenly painted with two additional coats of paint mixed with pure linseed-oil, of such color as may be directed; the tension members being, however, generally of lighter color than the compression members.

#### BRECTION.

- 1. The railroad company will take down the old bridge if any exists. It will furnish the lower falseworks, or supporting-trestles, only. The use of these falseworks by the contractor shall be construed as his approval of them.
- The contractor shall furnish all other staging (the plan and construction of which must be approved by the chief engineer), and shall erect and adjust all the ironwork complete.
- The contractor shall so conduct all his operations as not to impede the running of the trains or the operations of the road.
- 4. The contractor shall assume all risks of accidents to men or material during the manufacture and erection of the bridge.

ADDITIONAL STIPULATIONS.

The above specifications are approved.

Chief Engineer N. Y., L. E., & W. R.R.

The above specifications are accepted.

Contractor.

These specifications will be modified to suit the character of bridges here considered.

### SECTION 2.

# The Brunel Girder of Single System.

149. Take the form of girder shown in Fig. 16, Class I., article 49, where the end lengths of top chord are shorter than other segments. Bridge to have 2 equal parabolic double-bow girders, top and bottom chords of the same curvature. Floor carrying load attached by vertical struts and suspenders to bottom and top panel points, and in the plane of the axes of the two girders.

To compute the dimensions, let l = span in feet, h = height of the two equal parabolas composing each girder at the centre of span. Take the axis of x horizontal, that of y vertical; then the equation to the upper parabola, origin at top, is

$$x^2 = ay$$
.

But, with origin at left end, the equation is

$$y = \frac{2h}{l} \left( x - \frac{x^2}{l} \right). \tag{472}$$

If there are n equal panels, then the value of y at the r<sup>th</sup> panel point, where  $x = \frac{rl}{n}$ , becomes

$$y = \frac{2h}{n^2}r(n-r); (473)$$

and the whole height is

$$2y = \frac{4h}{r^2}r(n-r). (474)$$

By (473),

For 
$$r = 0$$
,  $y = 0$ ;  
 $r = 1$ ,  $y = 2h\frac{n-1}{n^2}$ ;  
 $r = 4$ ,  $y = 2h\frac{4n-16}{n^2}$ ;  
 $r = 4$ ,  $y = 2h\frac{4n-16}{n^2}$ ;  
 $r = 5$ ,  $y = 2h\frac{5n-25}{n^2}$ ; etc.

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and generally

$$\frac{1}{\cos^2 \alpha_r} = 1 + \frac{4h^2}{n^2 l^2} \epsilon_{i}^2. \tag{476}$$

VALUES OF 8, IN (476).

r	n=4	6	8	10	12	14	16	18	20	22	24
1 2 3 4 5 6	3 0	5 2	7 4 0	9 6 2	8. 4 0	13 10 6 2	15 12 8 4 0	17 14 10 6 2	19 16 12 8 4	21 18 14 10 6	23 20 16 12 8

In the same manner, we derive

$$\frac{1}{\cos^{2}\beta_{1}} = 1 + \frac{4h^{2}}{n^{2}l^{2}}(n-2)^{2}, \quad \frac{1}{\cos^{2}\beta_{3}} = 1 + \frac{4h^{2}}{n^{2}l^{2}}(n-10)^{2};$$

$$\frac{1}{\cos^{2}\beta_{2}} = 1 + \frac{4h^{2}}{n^{2}l^{2}}(n-6)^{2}, \quad \frac{1}{\cos^{2}\beta_{4}} = 1 + \frac{4h^{2}}{n^{2}l^{2}}(n-14)^{2};$$

$$\frac{1}{\cos^{2}\beta_{r}} = 1 + \frac{4h^{2}}{nl^{2}}\epsilon_{2}^{2}. \tag{477}$$

Values of  $\epsilon_2$  in (477).

r	n = 4	6	8	10	12	14	16	18	20	22	24
1 2 3 4 5 6	2	4 0	6 2	8 4 0	10 6 2	12 8 4 0	14 10 6 2	16 12 8 4 0	18 14 10 6 2	20 16 12 8 4	22 18 14 10 6

•	*-
1 2 3 4 5 6 7 8 9 10	7 3
11 12	

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 $\frac{l}{n\cos}$ 

Take  $q_1 = 18$  feet = extreme width of bridge.

q = 16 feet = width of floor.

 $\phi_i$  = inclination to plane of girder, of any hor diagonal.

Then

$$\frac{1}{\sin^2 \phi_1} = 1 + \frac{l^2}{18^2 n^2}.$$
 (4)

 $\frac{18}{\sin \phi_t}$  = length of horizontal diagonal supposed to from end to end of the transverse I floor beams.

150. To compute the Moments and Strains in C due to the Total Panel Weight, W+L, applied at Apex, Top and Bottom. — We have, from equation moments at the vertical sections through these apices,

$$M_r = \frac{W+L}{2n}l(n-r)r = \frac{W+L}{2n}k_4. \qquad (4)$$

Values of  $e_4$  in (480).

*	n = 4	•	8	10	12	14	16	18	20	21
1 2	3 4	5 8	7 12	9	11 20	13 24	15 28	17 32	19 36	2
3 4		9	15	21 24	27 32	33 40	39 48	45 56	5t 64	7
5		:		25	35 36	45 48	55 60	65 72	75 84	9
7 8 9		i			!	49	63 64	77 08 18	96 91	10 11 11
10									100	1:
12										

Owing to the peculiar form  $\left(0.5 + \frac{I}{2\epsilon}\right)$ , these values are easily written from a table of reciprocals.

# 151. Weights of these Wrought-Iron Chords.

W = unknown panel weight of bridge.

L = given panel weight of live load.

Q = allowed inch strain in top chord.

T = allowed inch strain in bottom chord, all in tons, say.

 $\frac{P}{O}$  = cross-section of top chord, square inches.

 $\frac{U}{T}$  = cross-section of bottom chord, square inches.

Volume of Segments of Top Chord, Cubic Inches.

$$\frac{12lH_1}{Qn\cos^2\alpha_1}, \quad \frac{24lH_2}{Qn\cos^2\alpha_2}, \quad \frac{24lH_4}{Qn\cos^2\alpha_3}, \quad \frac{24lH_6}{Qn\cos^2\alpha_4}, \text{ etc.}$$

Volume of Segments of Bottom Chord, Cubic Inches.

$$\frac{24lH_1}{Tn\cos^2\beta_1}, \quad \frac{24lH_3}{Tn\cos^2\beta_2}, \quad \frac{24lH_5}{Tn\cos^2\beta_3}, \quad \frac{24lH_7}{Tn\cos^2\beta_4}, \text{ etc.}$$

Weight of top chords,
$$\left. = \frac{5}{18} \times \frac{24^{l}}{nQ} \Sigma \frac{H}{\cos^{2}\alpha} \right.$$

$$= \frac{5}{18} \times \frac{24^{l}}{nQ} \times \frac{W + L}{h} \times \frac{ln}{4} \Sigma \frac{e_{4}}{\epsilon} \left( 1 + \frac{4h^{2}}{n^{2}l^{2}} e_{1}^{2} \right)$$

$$= \frac{5}{3} \times \frac{W + L}{Oh} \left( l^{2} \Sigma \frac{e_{4}}{\epsilon} + \frac{4h^{2}}{n^{2}} \Sigma \frac{e_{4}}{\epsilon} e_{1}^{2} \right), \quad (484)$$

by reason of (476) and (481).

In summing (484), it will be seen that only one of the two extreme panels is to be counted.

-											
**	00000910	0.555555	0-538462	0.529413	0.523810	0.530000	0.517241	0.515150	0.513514	0.512195	0,511711
q	0.571429	0.533333	0.571429 0.533333 0.531739	0.516139	0.512821	0.510638	0.509091	0.507937	0.507042	@.50632g)	0.505747
*			0.510139	0.510638)	0.507937	0-806339	0.505263	0.504505	0.503937	0.503497	0.503145
9					0.509042	0.505363	0.504202	0.503497	966eo5.0	0.500618	0.508336
00							0.503937	0.503145	0.503618	0.502249	o. gatgón
2									6.5005.0	o.sosoga	0.501793
8											0.501743
М		1.622333	1.171429 1.622222 2.098069	9. \$829 <b>4</b> 6	3.072368	3.564460	4.058390	4 553390	daeáta-S	5-545751	6.042795

VALUES OF  $\frac{64}{8}e_{1}^{2}$  TO BE USED IN SUMMING (484).

	***	ψ	80	10	13	71	16	18	04	**	*	
H	\$.4000	13.8889	26.3846	42.8824	63.3810	87.8500	116.3793	148.8788	185.3786	225 8780	270.3780	
- 61	۰	2 1333	8.3478	18.5806	32.8205	\$1.0638	73.3091	99-5556	129.8028	164.0506)	202 2090	_
*			۰	gorore (	8.1370	18.2278	32,3368 >	\$0 4505	72.5009	98.6854	128 8050	
9					۰	2,0311	8.0672	18,1259	32.1916	\$0.2618	72.3390	
40							۰	grors)	8.0419	16,0807	32.1255	
2										B.cotBe	8.0377	
2											٥	
м	\$-400p	18.1555	43.0803	84.1088	145.0760	4305-064	343-Bo54	489.1680	670.5830	Bys.og.28	11.57 3544	
			i									

We shall here assume that the ratio of the panel lest top chord to its least radius of gyration is 100, and the of iron can be exactly fitted to meet the required strain have, then, for the segments of the top chord, by our sp tions for columns with flat ends,

$$Q = \frac{4}{1 + \frac{100^2}{40000}} = 3.2 \text{ tons per square inch of section.}$$

These values of  $\Sigma_{\epsilon}^{\underline{e}_{\underline{d}}}$ ,  $\Sigma_{\epsilon}^{\underline{e}_{\underline{d}}}$   $e_{i}^{a}$ , and Q, put in (484), give,

Weight of top chords, in pounds, 
$$= \frac{W + L}{h}$$

$$(0.610120l^{3} + 0.70313.$$

$$(0.844908l^{2} + 1.05067.$$

$$(1.092744l^{2} + 1.40235.$$

$$(1.345284l^{2} + 1.75268.$$

$$(1.600192l^{2} + 2.10179.$$

$$(1.856490l^{2} + 2.45010.$$

$$(2.113692l^{2} + 2.79790.$$

$$(2.371521l^{2} + 3.14537.$$

$$(2.629797l^{2} + 3.49263.$$

$$(2.888413l^{2} + 3.83976.$$

$$(3.147290l^{2} + 4.18679.$$

Similarly we find, from (477) and (481),

to be summed as follows: --

Values of  $\frac{\epsilon_4}{\epsilon}$  in (485). Brace includes Numbers to be taken Twice.

•	*   *	•	<b>x</b> 0	10	18	14	18	18	80	22	<b>54</b>
0	<b>∂</b> ∞∞∞ ∘	0.555555 {	0.538462	0.529412	0.523810)	0.520000	0.517241	0.515152	0.513514	0.512195	0.511111
<b>w</b>		0.529412	0.517241 )	0.512195	0.509434	0.507692	767gc 0	9.50508	0.504950	0.504425	0.504000
2				0.510204	0.507246 )	0.505618	0.504587	0.503876	0.503356	0.502959	0.502646
-				-		0.505155	0.504000	0.503268	0.502762	0.502392	0.502110
0								0.503115	0.503538	0.502146	0.501859
H										0.502075	0.501754
M	7.300000	1.640523	2.111406	2.593418	3.080980	3-571775	4.064644	4-558943	5-054240	5.550309	6.046960

Values of  $\frac{e^4}{\epsilon_2}$  in (485) to be used Twice.

	-							9		8	
<u> </u>	*		XC	2	21	*1	91	201	<b>3</b>		<b>4</b> 2
H	2.4000	8.8888	19.3846	33.8824	52.3810	74.8800	101.3793	131.8788	166.3785	204.8780	247.3778
m		0	2.0690	8.1951	18.3396	32.4923	50.6494	72.8090	98.9702	129.1328	163.2690
N				0	2.0290	8.0899	18.1651	32.2481	50.3356	72.4261	98.5186
7						•	2.0160	8.0523	18.0994	32.1531	50.2110
0,					-			•	2,0101	8.0343	18.0669
H										0	2,0070
M	4.8000	17.7778	42.9078	84.1550	145.4999	**************************************	344.4194	489.9764	671.5876	893.9486	3000,8211

Since T = 5 tons, we have, from (485),

i

Weight of bottom chords, in pounds, 
$$= \frac{W + L}{h}$$
 (0.40000 $l^2 + 0.4000h^2$ ) 4 (0.546841 $l^2 + 0.65844h^2$ ) 6 (0.703802 $l^2 + 0.89390h^2$ ) 8 (0.864473 $l^2 + 1.12207h^2$ ) 10 (1.026993 $l^2 + 1.34722h^2$ ) 12 (1.190592 $l^2 + 1.57091h^2$ ) 14 (1.354881 $l^2 + 1.79385h^2$ ) 16 (1.519648 $l^2 + 2.01636h^2$ ) 18 (1.684746 $l^2 + 2.23863h^2$ ) 20 (1.850103 $l^2 + 2.46074h^2$ ) 22 (2.015653 $l^2 + 2.68264h^2$ ) 24

152. To find the Greatest Strains and the Weights in the Web System. — Calling L=0 in (481), and taking first differences, we find horizontal component of strain in any girder diagonal due to dead load, nW, thus:

$$\Delta H_{\mathcal{W}} = \frac{W}{h} \frac{\ln \Delta \frac{e_4}{\epsilon}}{4}.$$
 (486)

Values of  $\Delta_{8}^{24}$  to be used in (486).

24	-0.005364	-0.000855	-0.000320	-0.000149	-0.000067	-0.000012	+0.000038	+0.000103	+0.000216	+0.000499	+0.001747	-0.004165
83	-0.005866	-0.000928	-0.000341	-0.000150	-0.000054	+0.000017	+0.000096	+0.000226	+0.000538	+0.001904		-0.004558
03	-0.006472	-0.001013	-0.000362	-0.000144	-0.000025	+0.000080	+0.000232	+0.000581	+0.002092			160\$00'0-
18	-0.007215	-0.001113	-0.000379	-0.000123	+0.000030	+0.000229	+0.000629	+0.002319				-0.005633
16	-0.008150	-0.001231	-0.000385	-0.000063	+0.000202	+0.000676	+0.002597					-0.006354
14	-0.009362	-0.001363	-0.000355	+0.000108	+0.000711	+0.002946						-0.007315
18	-0.010989	-0.001497	-0.000204	169000-0+	+0.003387	-						-0.008612
10	-0.013283	-0.001557	+0.000434	+0.003934		,	. •					-0.010472
œ	-0.016723	-0.001112	+0.004498		_							-0.013337
•	-0.022223	+0.003921										-0.r 1830r
# #	-0.028571											-0.038571
x1+x_	۲ 1	<del>+</del> 3	6- 5	8-7	9 -01	12-11	14-13	16-15	18-17	20-19	22—21	M

These values of  $\Delta^{\frac{8}{4}}$  are the alternate first differences; the only ones required in this calculation, since the whole set of diagonals, Fig. 16, is strained equally with the whole set Y diagonals under the same uniform dead load and same liveload supposed to pass either way. And, moreover, we shall compute only weights of diagonals due to the greatest compressive strains developed in them; since, although they a alternately in compression and tension under liveload, the allowed unit strain in tension, 5 tons per square inch, is a much greater than that allowed in compression, that, if a dialonal can resist the compression, it certainly can resist the telesion coming upon it.

We shall, however, augment the cross-section of all gird-diagonals by eight-tenths of the section computed to resist the compressive strain, though not quite in accord with our gener specifications. Also, each girder diagonal is to be rigidly connected, at or near its centre, with the floor system, so the virtually the unsupported length of these struts is but half what it otherwise would be. In these ways we guard again lateral shocks received by the girder diagonals.

It may be noted here that the differences,  $\Delta \frac{\epsilon_4}{\epsilon}$ , in (486 would all vanish if both chords coincided with the parabolic curthroughout. Also, these differences, being negative, increathe counter strains, and diminish the main strains, due live loa contrary to what results when the chords are horizontal.

For the live load, nL, using the values of  $h_r$  in (475), at taking  $M_r$  from equation (64), we have

$$(H_L)_r = \frac{M_r}{h_r} = \frac{\frac{Ll}{2n^2} r(r+1)(n-r)}{\frac{2h}{n^2} l_r} = \frac{L}{h} \times \frac{l}{4} l_5 \qquad (48)$$

$$r(r+1)(n-r)$$

if 
$$\varepsilon_5 = \frac{r(r+1)(n-r)}{\varepsilon_r}$$
.

 $(H_L)_r$  = the horizontal component of chord strain at the foremost end of live load, due the same.

Also, from (68) and (475), we have

$$(H_L)_{r+1} = \frac{M_{r+1}}{h_{r+1}} = \frac{\frac{Ll}{2n^2}r(r+1)(n-r-1)}{\frac{2h}{n^2}e_{r+1}} = \frac{L}{h} \times \frac{l}{4}e_6 \quad (488)$$

if 
$$\varepsilon_6 = \frac{r(r+1)(n-r-1)}{\varepsilon_{r+1}}$$
.

 $(H_L)_{r+1}$  = horizontal component of chord strain due live load at one interval before its foremost end, and is simultaneous with  $(H_L)_r$  in (487);

$$\therefore \Delta H_L = \frac{L}{h} \times \frac{l}{4} (\epsilon_6 - \epsilon_5), \qquad (489)$$

which is the horizontal component of greatest strain on diagonals due live load alone, and made positive, and added to  $\Delta H_W$  also made positive, gives the total horizontal component of maximum diagonal strain, as thus expressed:

$$\Delta H = \frac{l}{4\hbar} \left\{ L(\epsilon_5 - \epsilon_6) - nW\Delta \frac{\epsilon_4}{\epsilon} \right\}. \tag{490}$$

Here also we require only the alternate values of  $\varepsilon_5$  and  $\varepsilon_6$ ; that is, those values which correspond to the instants when the foremost panel weight of live load is directly under the upper end of the Z member.

# CALCULATION OF THE WEIGHT OF BRIDGES.

•												
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	4	17 100	±1#	\$ 15	t							
		9. th	4 2	\$   12	#   #							
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	•	\$ 8	218	71 150	6 2 2	34 15						
		2 %	65	8 8	392 97	<b>%</b> \$	<b>5</b> 9					
	3	4,5	79	940 Opt	336 95	3 3	47 47					
	3	8 8	156 77	<b>Sor</b>	\$61 \$05	630	<b>5</b> 166	77 346				
	3	S - 56	55 141	300	148	611 045	95 S28	25. 70.				
	2	3 2	\$ \$	1 gg	151 g	gro 161	924 44	010	<b>8</b> . 30			
	3	Sig.	168	136s	156	720 759	792	7:28	\$\\\ \g			
	-	38 37	ğ   ğ	\$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$ \$	181 181	197	1188	181 181	1200	101 916		
	3	개병	193	167	호 lớ	ž %	191	167 167	960 001	11 71 713		
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		\$ 18	\$5# 12\$	570	217 239 239	209	1716 385	285	90 gr	2142	1900 000	\$61 98 <sup>E</sup> T
ı			\$   \$	3 Kg	55e 96g	1960 Odes	1584 287	279	1920	215	65r 0c51	97

VALUES	OF	<b>ε</b> <sub>5</sub>		<b>2</b> 6	IN	(490).
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<i>r</i>	×=4	6	8	10	12	14	16	18	20	22	<b>!</b> !
I	0.62857	0.57778	0.55518	0.54269	0.53480	o.5 <b>2</b> 936	0.52539	0.52236	0.51999	0.51806	0.5164
3		0.51765	0.52058	0.51686	0.51393	0.51179	0.51018	0.50896	0.50799	0.50721	0.50657
5			0.49475	0.50803	0.50827	0.50739	0.50651	0.50578	0.50516	0.50466	0.50424
7				0.48466	0.50242	0.50440	0.50444	0.50413	0.50378	0.50344	0.5031
9		1			0.47895	0.49923	0.50218	0.50276	0.50277	0.50263	0.5004
1							1		0.50167		_
3									0.49975		
5	ı	,							0.49463		
7	i								0.46939	0.49381	0.49844
9										0.46825	0 49317
2 2										l	0.46731
Σ	0.62857	1.09543	1.57051	2.05224	2.53837	3.02746	3.51859	4.01129	4.50511	4.99985	5-4953

Since the girder diagonals are to have their end bearings on the chord pins, and are to have their centres attached to the floor system, we shall take the ratio of whole length of diagonal to least radius of gyration = 100, and the allowed inch strain

$$Q_1 = \frac{4}{1 + \frac{100^2}{20000}} = \frac{8}{3}$$
 tons.

We have, then, after multiplying by 1.8, as above explained,

Cross-section of 1 diagonal = 
$$\frac{1.8\Delta H}{Q_r \cos \theta}$$
 square inches.

Volume of 1 diagonal 
$$=\frac{12l}{n} \times \frac{1.8\Delta H}{O_1 \cos^2 \theta}$$
 cubic inches.

Weight of 1 diagonal 
$$=\frac{5}{18} \times \frac{12l}{n} \times \frac{1.8\Delta H}{Q_1 \cos^2 \theta}$$
 pounds.

Weight of all girder diagonals, in pounds,
$$= 2 \times \frac{5}{18} \times \frac{12l}{n} \times \frac{3 \times 1.8}{8} \Sigma \frac{\Delta H}{\cos^2 \theta}$$

$$= \frac{L}{h} \left\{ \frac{9l^2}{8n} \Sigma (\epsilon_5 - \epsilon_6) + \frac{9h^2}{2n^3} \Sigma \epsilon_3^2 (\epsilon_5 - \epsilon_6) \right\}$$

$$- \frac{IV}{h} \left\{ \frac{9l^2}{8} \Sigma \Delta \frac{\epsilon_4}{8} + \frac{9h^2}{2n^2} \Sigma \epsilon_3^2 \Delta \frac{\epsilon_4}{8} \right\}, \quad (491)$$

from (478) and (490).

Placing the values of  $\Sigma(\varepsilon_5 - \varepsilon_6)$ ,  $\Sigma \varepsilon_3^2(\varepsilon_5 - \varepsilon_6)$ ,  $\Sigma \Delta \frac{\varepsilon_4}{\varepsilon}$ , and  $\Sigma \varepsilon_3^2 \Delta \frac{\varepsilon_4}{\varepsilon}$ , in (491), we find

Weight of girder diagonals, in pounds,

$$= \frac{W}{h} \begin{pmatrix} 0.032142l^2 + 0.39375h^2 \end{pmatrix} + \frac{L}{h} \begin{pmatrix} 0.176785l^2 + 2.16563h^2 \end{pmatrix} \begin{pmatrix} h \\ 4 \end{pmatrix} \\ \begin{pmatrix} 0.020589l^2 + 0.32779h^2 \end{pmatrix} \\ \begin{pmatrix} 0.015004l^2 + 0.26906h^2 \end{pmatrix} \\ \begin{pmatrix} 0.01781l^2 + 0.22622h^2 \end{pmatrix} \\ \begin{pmatrix} 0.009688l^2 + 0.19425h^2 \end{pmatrix} \\ \begin{pmatrix} 0.008229l^2 + 0.17048h^2 \end{pmatrix} \\ \begin{pmatrix} 0.007148l^2 + 0.15133h^2 \end{pmatrix} \\ \begin{pmatrix} 0.006326l^2 + 0.13892h^2 \end{pmatrix} \\ \begin{pmatrix} 0.005128l^2 + 0.11352h^2 \end{pmatrix} \\ \begin{pmatrix} 0.004686l^2 + 0.10456h^2 \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0.257592l^2 + 86.06221h^2 \end{pmatrix} \begin{pmatrix} 22 \\ 22 \\ 23 \end{pmatrix} \begin{pmatrix} 22 \\ 23 \end{pmatrix} \begin{pmatrix} 1 \\ 24 \\ 24 \end{pmatrix} \begin{pmatrix}$$

Weight of girders without head system, in pounds,

$$= \frac{IV}{h} \left( 1.042262l^2 + 1.49688h^2 \right) + \frac{L}{h} \left( 1.186905l^2 + 3.26876h^2 \right) \left[ 4.1412338l^2 + 2.03690h^2 \right) \\ \left( 1.811550l^2 + 2.56531h^2 \right) \left( 2.221543l^2 + 3.10097h^2 \right) \\ \left( 2.636873l^2 + 3.64326h^2 \right) \\ \left( 3.055311l^2 + 4.19149h^2 \right) \\ \left( 3.475721l^2 + 4.74308h^2 \right) \\ \left( 3.897495l^2 + 5.30065h^2 \right) \\ \left( 4.320203l^2 + 5.85439h^2 \right) \\ \left( 4.743644l^2 + 6.41402h^2 \right) \\ \left( 5.167629l^2 + 6.97399h^2 \right) \right] \left( 1.186905l^2 + 3.26876h^2 \right) \\ \left( 1.597142l^2 + 6.86007h^2 \right) \\ \left( 2.017398l^2 + 11.62470h^2 \right) \\ \left( 2.440634l^2 + 17.58571h^2 \right) \\ \left( 2.865157l^2 + 24.74715h^2 \right) \\ \left( 3.290360l^2 + 33.10947h^2 \right) \\ \left( 3.715974l^2 + 42.67217h^2 \right) \\ \left( 4.141874l^2 + 53.43638h^2 \right) \\ \left( 4.567955l^2 + 65.40090h^2 \right) \\ \left( 4.994170l^2 + 78.56577h^2 \right) \\ \left( 5.420535l^2 + 92.93164h^2 \right) \\ \left$$

153. The floor; to be made of  $2\frac{1}{2}$ -inch oak planks weighing 52 pounds per cubic foot. Width = q = 16 feet in (432).

... Weight of floor = 
$$F = \frac{2.5}{12} \times 16 \times 52l = \frac{520}{8}l$$
 pounds. (

154. The joists of oak; longitudinal, spaced 2 feet betw centres, and consequently 9 in number, all of equal size.

By (431) and (432), we have

Depth of joists = 
$$b^2 = \left\{ \frac{9 \times 10 \times 2l}{16 \times 10000n^2} (F + 2000nL) \right\}^{\frac{6}{5}}$$
 inches, (...

calling B = 10,000 pounds = breaking inch strain for f = 10 = factor of safety, g = 2 feet = distance betw centres of joists.

And, from (433),

Weight of 9 joists = 
$$J = \frac{9 \times 52l}{144} \left\{ 0.001125 \frac{l}{n^2} (F + 2000nL) \right\}^{\frac{3}{2}}$$
  
=  $\frac{13}{4} l \left\{ 0.001125 \frac{l}{n^2} (F + 2000nL) \right\}^{\frac{3}{2}}$  pounds. (.

155. The wrought-iron I floor beams, n-1 in nun (single or in pairs, according to live load, as shown below), a bearing I panel weight of live load, of floor, and of jo besides its own weight, which is provided for as explained article 124.

Also, the I-beams must bear the longitudinal strain du wind pressure, and to initial strain on the horizontal diago inserted between them in each panel.

Taking the proportions of the beam's section as in ar 124, and calling  $\frac{B}{f} = 10,000$  pounds = allowed inch strain wrought-iron beams, (412) gives

Depth of I-beam = 
$$d_z = 3.80122 \left\{ \frac{(F + J + 2000nL)18}{10000n} \right\}^{\frac{1}{4}}$$
, (in inches,

Cross-section of I-beam, from (413), is

$$S = \frac{107}{1200} d_2^2 \text{ square inches.}$$

Weight of 
$$(n-1)$$
I-beams, due to load, pounds,
$$=P=15.46068 \times \frac{5}{18}$$

$$\times 18(n-1) \left\{ \frac{18(F+J+2000nL)}{10000n} \right\}^{\frac{1}{2}}$$

$$=77.3034(n-1)\left\{\frac{18(F+J+2000nL)}{10000n}\right\}^{\frac{2}{3}}, (496)$$

by reason of (414).

156. Or, in order to avoid the complicated expressions for weight of joists in terms of L and l, we may proceed as follows:—

By equation (408),

Weight of floor = F = ulqt pounds,

$$\therefore uqt \frac{l}{n} + 2000L = \text{panel weight of floor and live load,}$$

$$\frac{g}{g}\left(uqt\frac{l}{n} + 2000L\right)$$
 = uniform load on each panel length of joist.

Putting this weight for *lw* in equation (52), we have, for each panel length of joist,

Moment due floor and live load, 
$$M = \frac{1}{8} \frac{g}{g} \left( uqt \frac{l}{n} + 2000L \right) \frac{12l}{n}$$
  
=  $\frac{1}{8} B_1 b d^2$ , (497)

by (161);  $B_{\rm r}$  being the allowed inch strain.

Now we will take  $bd^2 = \frac{l}{n}S$ ;

$$\therefore d = \frac{l}{n}$$

(d in inches, l in feet),

$$\therefore S = \frac{n}{l}bd^2 \text{ square inches.}$$
 (498)

That is, the cross-section, S, of a joist is taken equal to  $\frac{n}{l}$  times its breadth, b, multiplied by the square of its depth, d. Then, calling g = 2 feet, q = 16 feet, u = 52 pounds,  $t = \frac{2.5}{12}$  feet,  $B_1 = 1,000$  pounds per square inch for oak, equation (497) becomes, after reducing,

$$S = 0.001125 \left( \frac{520}{3} \frac{l}{n} + 2000L \right) \text{ square inches,}$$
 (499)

$$\therefore J = \frac{9 \times 5^{2} l}{144} S = 0.00365625 \left( \frac{5^{20}}{3} \frac{l}{n} + 2000 L \right) l \text{ pounds.} \quad (500)$$

$J = 7.3125 Ll + l^2$	$\frac{0.63375}{n}$ pound,	n
	0.1584375 pound,	4
	0.1056250 pound,	6
:	0.0792187 pound,	8
	0.0633750 pound,	10
	0.0528125 pound,	I 2
	0.0452679 pound,	14
	0.0396094 pound,	16
	0.0352083 pound,	18
	0.0316875 pound,	20
	0.0288068 pound,	22
	0.0264063 pound.	24

which are the weights of the 9 joists for oak.

157. If, instead of wood, we use wrought-iron I-beams to support the floor and load, we may assume the beam's cross-section to have such proportions (see manufacturers' tables) that

$$I = \frac{l}{10n} Sd, \tag{501}$$

From (52) and (187),

$$M = \frac{1}{8} \left( \frac{F + J_1 + 2000nL}{n} \right) 12q_1 = \frac{2B_1I}{d}.$$
 (504)

Take  $B_1 = 10,000$  pounds per square inch.

 $q_1 = 18$  feet = entire length of beam.

We shall assume the cross-section of each transverse I-beam to be such that

$$I = 2Sd (505)$$

whether the beam be rolled or made up of plate and angle iron. Therefore, from (504) and (505),

$$S = 0.000675 \left( \frac{520}{3} \frac{l}{n} + \frac{6Ll}{n} + \frac{0.52l^2}{n^2} + 2000L \right). \quad (506)$$

Weight of (n - 1) transverse I-beams due load, in pounds,

$$= (n-1) \times \frac{5}{18} \times 12 \times 18S$$

$$= 0.0405(n-1) \left( \frac{520l}{3n} + \frac{0.52l^2}{n^2} + \frac{6Ll}{n} + 2000L \right) \quad (507)$$

$$= \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & 5.2650 + \begin{array}{|c|c|c|c|c|} \hline & 0.003949 + Ll \\ \hline & 5.8500 & 0.002925 & 0.20250 & 405 & 6 \\ \hline & 6.1425 & 0.002303 & 0.21262 & 567 & 8 \\ \hline & 6.3180 & 0.001895 & 0.21870 & 729 & 10 \\ \hline & 6.4350 & 0.001609 & 0.22275 & 891 & 12 \\ \hline & 6.5186 & 0.001397 & 0.22564 & 1053 & 14 \\ \hline & 6.5812 & 0.001234 & 0.22781 & 1215 & 16 \\ \hline & 6.6300 & 0.001105 & 0.22950 & 1377 & 18 \\ \hline & 6.6690 & 0.001000 & 0.23085 & 1539 & 20 \\ \hline & 6.7009 & 0.000914 & 0.23195 & 1701 & 22 \\ \hline & 6.7275 & 0.000841 & 0.23287 & 1863 & 24 \\ \hline \hline \end{array}$$

159. In order that the wind pressure may be a function of the height, h, of the girders, as it manifestly ought to be, we will assume, for wind pressure against these highway bridges, 50 pounds per square foot of actual vertical surface presented by both girders, estimated at  $\frac{100}{l} \times \frac{h}{2}$  square feet, to the running-foot of bridge. Therefore

Wind pressure per linear foot =  $2500\frac{h}{l}$  pounds.

Wind pressure per panel length  $= W_1 = 2500 \frac{h}{n}$  pounds.

No account is here taken of vertical wind force.

Let the strains due this horizontal force of wind be provided for along the floor system which is placed midway between the top and bottom chords (that is, insert horizontal diagonals in each panel between the transverse I-beams), and increase the cross-section already found for these transverse beams by an amount required by the wind force; and let the longitudinal strain due wind be taken up by 2 longitudinal wrought-iron bars, or channels, extending the entire length of bridge, securely attached to the outside of the outer longitudinal floor beams, to the top of every transverse beam, and to each girder diagonal, as already explained.

These two longitudinal bars or channels, and the two outside floor beams, are to be made continuous throughout, and capable of resisting either tension or compression. It will be noticed, that the floor and load use only one-half the capacity of these two outside longitudinal floor beams; also, that all floor beams, longitudinal and transverse, are, from the manner of their loading, unable to deflect horizontally.

160. The Horizontal Diagonals, 2n in Number. — To provide for travelling gusts of wind, we shall here assume that

this panel pressure,  $W_1 = 2.500 \frac{h}{n}$  pounds, is a uniform live load.

Therefore the strains upon the diagonals are given in Fig. 112 if we put  $W_1$  in the place of L, and make W = 0.

But, since we must provide for this wind pressure coming upon either side of the bridge, it is plain that all horizontal diagonals must be "mains," and the two in any panel equal in size. We must, therefore, take  $\frac{n}{2}$  terms of the following series four times:

Strain on horizontal diagonals due wind

$$= \frac{IV_{I}}{2n\sin\phi_{I}} \left\{ n(n-1), (n-1)(n-2), (n-2)(n-3), \dots \frac{n}{2} \text{ terms} \right\}, (508)$$

in same denomination as  $W_{\rm r}$ .

Take the inch strain  $T_1 = 15,000$  pounds.

Weight of horizontal diagonals, pounds,

$$= 4 \times \frac{12q_1}{\sin \phi_1} \times m \times \frac{W_1}{2nT_1 \sin \phi_1} \begin{cases} n(n-1) + (n-1)(n-2) \\ + (n-2)(n-3) \\ + (n-3)(n-4) \end{cases}$$

$$= \frac{24mq_1 W_1}{nT_1 \sin^2 \phi_1} (\frac{7}{24}n^3 - \frac{1}{6}n)$$

$$= \frac{5}{6}h \left(7n - \frac{4}{n}\right) \left(1 + \frac{e^{l^2}}{324n^2}\right), \tag{509}$$

since  $m = \frac{5}{18}$  pounds per cubic inch for wrought-iron,  $q_i = 18$  feet = length of transverse I-beams.

$$\frac{1}{\sin^2\phi_1}=1+\left(\frac{l}{nq_1}\right)^2,$$

by (479).

From (509), we find Weight of horizontal diagonals, pounds,

			n
= h	$22.500 + hl^2$	0.0043403	4
	34-444	0.0029531	6
ł	46.250	0.0022304	8
	58.000	0.0017901	10
	69.722	0.0014944	12
	81.429	0.0012823	14
	93.125	0.0011225	16
	104.815	0.0009985	18
	116.500	0.0008989	20
į	128.182	0.0008174	22
	139.861	0.0007494	24

161. The Horizontal Struts; that is, in this Case, the Quantity of Iron to be added to the Transverse I-Beams by Reason of Wind Pressure. — If we divide the terms of equation (508) by 15,000, we shall have the cross-sections of the horizontal diagonals in square inches. And, if each of these sections be multiplied by  $10,000 \sin \phi_1$ , the product will be the longitudinal pressure brought upon the end of any transverse I-beam by one horizontal diagonal. Now, by our specifications, the pressure so brought upon these horizontal struts by the horizontal diagonals attached to the end of each is the end pressure to be provided for in these struts or I-beams. We therefore have

Longitudinal pressure upon end of transverse I-beams

$$= \frac{W_1}{3n} \left\{ \begin{bmatrix} n(n-1) \\ +(n-1)(n-2) \end{bmatrix}, \begin{bmatrix} (n-1)(n-2) \\ (n-2)(n-3) \end{bmatrix}, \text{ etc.} \right\}$$

$$= \frac{2W_1}{3n} [(n-1)^2, (n-2)^2, (n-3)^2, \text{ etc.}]. \quad (510)$$

These I-beams being unable to deflect horizontally, and having considerable depth, we may take for them, under this wind pressure, the unit strain,

$$Q_{2} = \frac{\frac{8000}{1 + \frac{(12q_{1})^{2}}{20000 \times \frac{2.5?}{n}}} = \frac{\frac{8000}{1 + 0.933127} \text{ pounds per square inc}$$

where  $\frac{2.5l}{n}$  is put for the square of the radius of gyration a an axis normal to web of beam.

Hence the areas of sections to be added to the I-beams reason of wind, are

$$S = \frac{2W_1}{3nQ_2} [(n-1)^2, (n-2)^2, (n-3)^2, \text{ etc.}]. \quad (511)$$

Taking  $m = \frac{5}{18}$ ,  $q_1 = 18$ ,  $W_1 = 2,500\frac{h}{n}$ , we find

Weight of iron to be added to transverse I-beams, on account of in pounds,

$$= 4 \times 12q_{1} \times \frac{mW_{1}}{3nQ_{2}} \begin{cases} (n-1)^{2} + (n-2)^{2} + (n-1)^{2} \\ \dots \left(\frac{n}{2}-1\right) \text{ terms } + \frac{n^{2}}{8} \end{cases}$$

$$= \frac{2mq_{1}W_{1}}{3Q_{2}} (7n^{2} - 12n + 2) \quad (n \text{ even}),$$

$$= h \left(\frac{25}{24} + 0.972\frac{n}{7}\right) \left(7n - 12 + \frac{2}{n}\right) \qquad (512)$$

$$= h \left(\frac{31.597}{46.111} + \frac{176.91}{344.09} + \frac{6}{8} + \frac{176.91}{60.625} + \frac{6}{565.71} + \frac{10}{10} + \frac{175.173}{89.733} + \frac{841.75}{1172.18} + \frac{12}{14} + \frac{1172.18}{118.866} + \frac{1172.18}{11996.49} + \frac{11}{18} + \frac{118.866}{118.866} + \frac{1996.49}{118} + \frac{118.866}{118.868} + \frac{1172.18}{118.868} + \frac{11$$

resist Wind Force. — The strains generated in these chords by the panel pressure of wind,  $W_1$ , are given in Fig. 112 if, for  $N = \frac{W + L}{2nh}l$ , we put  $N = \frac{W_1l}{2nq_1}$ , thus:

Maxima chord strains in horizontal system

$$= \frac{W_i l}{2nq_i} [(n-1), 2(n-2), 3(n-3), \text{ etc.}], \quad (513)$$

for each chord, since the wind may act on either side of the bridge.

Now, since these strains will be sometimes in tension and sometimes in compression, these wind chords must be constructed so as to resist either kind of strain. Then, of course, a cross-section sufficient for the greatest compressive strain will be ample for the maximum tensile strain.

And, because these chords are to be securely attached to the girder diagonals, to the outside longitudinal I-beams whose strength is only one-half taxed in supporting the floor and live load, and to the transverse I-beams, we shall take

$$\frac{\text{unsupported length}}{\text{radius of gyration}} = 100,$$

as in case of the top chords, article 151; also, call the ends fixed. The axis of gyration is normal to the plane of girder, since the floor prevents these struts from deflecting horizontally. Then

$$Q = 3.2 \text{ tons} = 6400 \text{ pounds} = \text{allowed inch strain.}$$

Cross-section of wind chords for each panel

$$= \frac{W_1 l}{2nq_1 Q} \left\{ n-1, \ 2(n-2), \ 3(n-3), \dots \frac{n}{2} \left(n-\frac{n}{2}\right) \right\}. \tag{514}$$

24

Weight of wind chords, in pounds,

$$= 4 \times \frac{5}{18} \times \frac{12l}{n}$$

$$\times \frac{lV_1 l}{2nq_1 Q} \left\{ (n-1) + 2(n-2) + 3(n-3), \dots \frac{n}{2} \text{ terms} \right\}$$

$$= 0.006028164 \left( 2 + \frac{3}{n} - \frac{2}{n^2} \right) h l^2 \qquad (515)$$

$$= h l^2 \begin{vmatrix} 0.0158239 & n = 4 \\ 0.0147353 & 6 \\ 0.0147353 & 6 \\ 0.0137442 & 10 \\ 0.0134800 & 12 \\ 0.0132866 & 14 \\ 0.0130238 & 18 \\ 0.0129304 & 20 \\ 0.0128534 & 22 \end{vmatrix}$$

163. The vertical supports for the transverse floor beams and their load must also resist the moment due that part of the wind pressure which acts upon the chords and girder diagonals.

1st, The total weight to be upheld by each of these vertical struts and hangers is the  $(2n)^{th}$  part of the sum of the weights of the live load, the floor, the longitudinal I-beams, the transverse I-beams taken  $\frac{n}{n-1}$  times, the horizontal diagonals in

the floor system, and the wind chords; all of which have now been found in terms of l and L. Call this weight  $\varepsilon_n$  pounds on each strut and on each hanger, n being the number of panels. Each strut, of course, transmits the load,  $\varepsilon_n$ , to the panel point below; while each hanger or suspender transmits the load,  $\varepsilon_n$ , to the alternate panel points above.

For the struts, we may take

$$Q_3 = \frac{8000}{1 + \frac{100^2}{20000}} = 5333 \text{ pounds}$$

as the allowed inch strain in compression.

 $\begin{array}{c} :. \quad \text{Cross-section of a strut due} \\ \text{vertical forces} \end{array} \right\} = S = \frac{\varepsilon_n}{Q_3} \text{ square inches.}$  (516)

Weight of all struts due vertical forces 
$$= 2 \times \frac{5}{18} \times \frac{\epsilon_n}{Q_3} \times 12 \Sigma y$$
$$= 0.000208 \frac{1}{3} \left( n - \frac{4}{n} \right) h \epsilon_n \text{ pounds,}$$
 (517)

since, from (473), for lower parabola,

$$\sum y = \frac{2h}{n^2} \left\{ (2n - 4) + (4n - 16) + (6n - 36) \dots \left( \frac{n}{2} - 1 \right) \text{ terms} \right\}$$
$$= \frac{1}{6} h \left( n - \frac{4}{n} \right).$$

Similarly, for the suspenders, take  $T_1 = 6,000$  pounds = the allowed inch strain in tension.

Cross-section of suspender 
$$S_i = \frac{\epsilon_n}{T_i}$$
 square inches. (518)

Weight of all suspenders due 
$$= 2 \times \frac{5}{18} \times \frac{\epsilon_n}{T_1} \times 12 \Sigma y$$
  
vertical forces  $= 0.000i8\dot{5}\left(n + \frac{2}{n}\right)\hbar\epsilon_n$  pounds, (519)

since, for the upper parabola,

$$\sum y = \frac{2h}{n^2} \left\{ (n-1) + (3n-9) + (5n-25) \dots \frac{n}{2} \text{ terms} \right\}$$
$$= \frac{1}{6} h \left( n + \frac{2}{n} \right).$$

2d, We shall assume, that, of the wind pressure, 125 pounds act upon each running-foot of the two top chords and of the two bottom chords, tending to rotate each girder about its longitudinal axis. We may note, that, in general, these forces acting upon one chord will be nearly balanced by the forces acting upon the other; but in certain cases a gale may strike one chord and not the other.

Acting, then, in lines normal to the plane of each girder, at each panel point or apex, is the pressure of  $62.5 \times \frac{2l}{n} = 125\frac{l}{n}$  pounds, with the lever arm, y, causing a moment, at the wide end of the strut or suspender, of  $125\frac{l}{n}y$  foot-pounds. We take no account here of the fact that each end segment of the top chord is only about one-half the length of any other.

Let these struts and suspenders, acting also as lateral braces to the chords where there is no lateral head system, have a breadth of effective base equal to  $\frac{1}{10}y$ . The broad end of the suspender is to be attached to the top chord and head lateral strut whenever it would obstruct unduly the roadway below. Otherwise, and in all cases where head laterals are wanting, the suspender has its broad end securely bolted to the transverse I-beam in the floor system.

Then, if  $S_2$  = cross-section of the two members or flanges of each strut or suspender, we have, at the broad end of each, this equality of moments,

$$125 \frac{l}{n} y = \frac{1}{2} S_2 \times \frac{1}{10} y B_1,$$

$$S_2 = \frac{2500l}{nB_1} = \frac{75}{170} \frac{l}{\mu} \text{ square inches,}$$
 (520)

if  $B_r = \frac{1}{2}(5,333 + 6,000) = 5,667$  pounds = allowed inch strain in bending.

It will be noticed that the cross-section,  $S_2$ , is uniform throughout the member if the two flanges meet at one end, as we shall assume they do, and shall illustrate in specifications.

We have, then,

Weight of verticals required to resist bending-moment due wind, in pounds,

$$= 2 \times \frac{5}{18} \times \frac{75}{170} \frac{l}{n} \times 12 \sum y = 0.9803922 \left(1 - \frac{1}{n^2}\right) hl, (521)$$

since now

$$\sum y = \frac{2h}{n^2} [(n-1) + 2(n-2) + 3(n-3) \dots (n-1) \text{ terms}]$$

$$= \frac{1}{8} h \left( n - \frac{1}{n} \right).$$

From (517), (519), and (521), we find

Weight of all vertical supports, in pounds,

$$= \hbar \left\{ \epsilon_n \left( 0.000393518n - \frac{0.000462963}{n} \right) + 0.9803922 \left( 1 - \frac{1}{n^2} \right) \ell \right\}$$
(522)
$$= \hbar \left\{ (0.001138Ll + 1.5174L + 0.000000735l^3 + 0.00002466l^2 + 0.95447l + 0.0031) \right\}$$
(6.001188Ll + 2.3765L + 0.000000673l^3 + 0.00001716l^2 + 0.99024l + 0.0081) \ (0.001206Ll + 3.2154L + 0.000000632l^3 + 0.00001306l^2 + 1.00373l + 0.0152) \ (0.001214Ll + 4.0464L + 0.000000604l^3 + 0.00001052l^2 + 1.01035l + 0.0244) \ (0.001218Ll + 4.8733L + 0.00000584l^3 + 0.00000880l^2 + 1.01470l + 0.0358) \ (0.001221Ll + 5.6980L + 0.000000570l^3 + 0.00000756l^2 + 1.01758l + 0.0494) \ (0.001223Ll + 6.5211L + 0.000000559l^3 + 0.0000062l^2 + 1.01988l + 0.0650) \ (0.001223Ll + 7.3434L + 0.000000550l^3 + 0.00000589l^2 + 1.02172l + 0.0828) \ (0.001225Ll + 8.9861L + 0.000000537l^3 + 0.00000483l^2 + 1.02487l + 0.1028) \ (0.001225Ll + 8.9861L + 0.000000537l^3 + 0.00000483l^2 + 1.02487l + 0.1249) \ 22

 $(0.001226Ll + 9.8069L + 0.000000531l^3 + 0.00000443l^2 + 1.02626l + 0.1491)$ 

since, for  $\varepsilon_n$  in (522), we have

 $\begin{aligned} & \epsilon_4 = 0.78037Ll + 1040.5L + 0.016908l^2 + 22.5441l + 0.002521hl^2 + 5.843h + 10.69\frac{h}{l}, \\ & \epsilon_6 = 0.52025Ll + 1040.5L + 0.007515l^2 + 15.0290l + 0.001474hl^2 + 6.030h + 17.69\frac{h}{l}, \\ & \epsilon_8 = 0.39019Ll + 1040.5L + 0.004227l^2 + 11.2720l + 0.001022hl^2 + 6.184h + 24.58\frac{h}{l}, \\ & \epsilon_{10} = 0.31215Ll + 1040.5L + 0.002705l^2 + 9.0176l + 0.000777hl^2 + 6.268h + 31.43\frac{h}{l}, \\ & \epsilon_{12} = 0.26013Ll + 1040.5L + 0.001879l^2 + 7.5147l + 0.000624hl^2 + 6.322h + 38.26\frac{h}{l}, \\ & \epsilon_{14} = 0.22297Ll + 1040.5L + 0.001380l^2 + 6.4412l + 0.000520hl^2 + 6.359h + 45.08\frac{h}{l}, \\ & \epsilon_{16} = 0.19509Ll + 1040.5L + 0.001057l^2 + 5.6360l + 0.000446hl^2 + 6.387h + 51.90\frac{h}{l}, \\ & \epsilon_{18} = 0.17342Ll + 1040.5L + 0.000835l^2 + 5.0098l + 0.000389hl^2 + 6.407h + 58.71\frac{h}{l}, \\ & \epsilon_{20} = 0.15608Ll + 1040.5L + 0.000676l^2 + 4.5088l + 0.000346hl^2 + 6.424h + 65.53\frac{h}{l}, \\ & \epsilon_{22} = 0.14189Ll + 1040.5L + 0.000559l^2 + 4.0989l + 0.000311hl^2 + 6.437h + 72.34\frac{h}{l}, \\ & \epsilon_{24} = 0.13006Ll + 1040.5L + 0.000676l^2 + 4.5088l + 0.000311hl^2 + 6.437h + 72.34\frac{h}{l}, \\ & \epsilon_{24} = 0.13006Ll + 1040.5L + 0.000470l^2 + 3.7574l + 0.000282hl^2 + 6.448h + 79.15\frac{h}{l}, \end{aligned}$ 

which, as above defined, is the weight upheld vertically by each support.

It will be observed, that, in all terms involving  $h^2$  in the expression for weight of vertical supports, equation (522),  $h^2$  has been replaced by  $\frac{1}{5}hl$ . This substitution is simply for convenience, and, being made in these small terms only, does not practically affect the accuracy of our resulting equations, while we are hereby relieved of higher powers of h than the second, in the value of IV.

164. As additional security against deflection, out of the plane of the girder, by the top chord, we shall insert a system of head lateral bracing between the two top chords where the

height is sufficient. For this purpose, the two top chords are the flanges of a great longitudinal strut or column, whose tendency to deflect laterally must be overcome by this head web system of diagonals and struts.

Suppose the moment at the centre of this system be that due to  $\frac{1}{4}W_1$  acting upon each panel length, or to  $\frac{1}{2}W_1$  acting upon the windward side at each joint of the windward top chord; that is, by (480), where now we must put  $\frac{1}{2}n$  for n, and  $\frac{1}{4}n$  for r, and  $2.500\frac{h}{n}$  pounds for  $W_1$ , we have

Moment at centre = 
$$M = \frac{1}{16}W_1 ln = 78.125 hl$$
;

and the longitudinal horizontal strain,

$$H = \frac{M}{q_1} = 78.125 \frac{hl}{q_1}$$
 pounds at centre,

which in each flange may be considered to decrease uniformly to the ends, as is practically the case with that part of the strain due to bending-moment in a pillar.

Therefore, for each double panel length,

$$\Delta H = H \times \frac{1}{\frac{1}{4}n} = 312.5 \frac{hl}{nq_1};$$

requiring each diagonal tie to resist

$$\frac{312.5}{\cos \alpha \cos \phi_2} \times \frac{hl}{nq_1} \text{ pounds,}$$

and to have a cross-section

$$S = \frac{312.5hl}{15000\cos u \cos \phi_2 n q_1} = \frac{0.0208 \frac{1}{3}hl}{n q_1 \cos \phi_2} \text{ square inches,}$$
 (523)

since  $\cos \alpha$  may, for these central panels receiving the head system, be put = 1 without practical error.

Length of each head diagonal = 
$$\frac{2l}{n\cos\phi_2}$$
 practically.

Weight of 
$$2\left(\frac{n}{2}-3\right)$$
 wrought-iron head diagonals, in pounds,

$$= 2\left(\frac{n}{2} - 3\right) \times \frac{5}{18} \times \frac{12 \times 2l}{n\cos\phi_2} \times \frac{0.0208\frac{1}{3}hl}{18n\cos\phi_2}$$

$$= 0.007716\frac{n - 6}{n^2} \times \frac{hl^2}{\cos^2\phi_2}$$

$$= 0.007716(n - 6)\left(\frac{hl^2}{n^2} + 81h\right) (524)$$

$$= h \begin{vmatrix} - & + hl^2 & - & 4 \\ 0 & 0 & 0 & 6 \\ 1.25 & 0.0002411 & 8 \\ 2.50 & 0.0003086 & 10 \\ 3.75 & 0.0003215 & 12 \\ 5.00 & 0.0003149 & 14 \\ 6.25 & 0.0003014 & 16 \\ 7.50 & 0.0002857 & 18 \\ 8.75 & 0.0002701 & 20 \\ 10.00 & 0.0002551 & 22 \\ 11.25 & 0.0002411 & 24 \end{vmatrix}$$

since

$$q_1 = 18$$
 feet, and  $\frac{1}{\cos^2 \phi_2} = 1 + \left(\frac{18n}{2l}\right)^2 = 1 + \tan^2 \phi_2$ .

If we multiply S in (523) by  $2 \times 10,000 \cos \phi_2 \tan \phi_2$ , we have, by our specifications, the end pressure brought by each pair of diagonals upon the end of each head strut; and, calling the inch strain in compression on these struts 2,500 pounds, we have the cross-section of each head strut, in square inches,

$$S = \frac{312.5 \times 2 \times 10000hl \tan \phi_2}{15000 \times 2500nq_1} = \frac{1}{12}h; \quad (525)$$

h being in feet, and  $\tan \phi_2 = \frac{nq_1}{2l}$ .

Weight of  $\left(\frac{n}{2}-2\right)$  head struts, in pounds,

$$= \left(\frac{n}{2} - 2\right) \times \frac{5}{18} \times 12 \times 18 \times \frac{h}{12} = 5h\left(\frac{n}{2} - 2\right) \quad (526)$$

h in feet.

Since we have already provided, in the floor system, for the whole bending-force of the wind, and are now simply stiffening the head system laterally as a column, or to meet adjustment strains, and strains due to imperfections in workmanship, it will manifestly suffice if we call the additional chord strain in each segment of top chord within the head system equal to

$$\Delta H = 312.5 \frac{hl}{nq^1} = 17.36 \frac{1}{9} \frac{hl}{n}$$
 pounds.

And, as 6,400 pounds is the allowed inch strain in top chords, we have

Cross-section to be added to each segment of top chord due to strain on head diagonals, square inches,

$$= S = 0.00271267 \frac{hl}{n}. \tag{527}$$

Weight of added iron in  $\left(\frac{n}{2}-3\right)$  of the central double panels, for top chords, in pounds,

$$= 2\left(\frac{n}{2} - 3\right) \times \frac{5}{18} \times \frac{12 \times 2l}{n} S = 0.0361689 \left(\frac{\frac{1}{2}n - 3}{n^2}\right) h l^2 (528)$$

$$= hl^{2}$$

$$- \frac{4}{0}$$

$$0.0005651$$

$$0.0007234$$

$$0.0007535$$

$$0.0007381$$

$$0.0007064$$

$$0.0006698$$

$$18$$

$$0.0006329$$

$$0.0005978$$

$$22$$

$$0.0005651$$

$$24$$

165. To find the Necessary Amount of Material for the Triangular Web System of Latticed Struts or Columns.

Let l' = length of strut.

d = effective width of strut.

A = area of both flanges in section normal to axis of strut, in square inches.

 $A_1$  = area of diagonals in the same section normal to axis of strut, and not to its own axis, in square inches.

 $\theta = 45^{\circ} = inclination of diagonal to axis of strut.$ 

 $B_1$  = allowed inch strain, both in flanges and diagonals, in the present case.

Then, moment at centre,

$$M = \frac{1}{4}AdB_1.$$

Longitudinal flange strain at centre =  $H = \frac{M}{d} = \frac{1}{2}AB_{1}$ .

Now, since H decreases uniformly from the centre to the ends, at least practically,

$$\Delta H = \frac{d}{\frac{1}{2}l'}H = \frac{AdB_{t}}{l'},$$

which is the longitudinal component of diagonal strain.

And

$$\Delta H + \cos \theta = \frac{AdB_i}{l'\cos \theta} = \text{strain on diagonal};$$

$$\therefore A_{l} = \frac{Ad}{l'\cos^{2}\theta} = \frac{2Ad}{l'}, \qquad (529)$$

$$\frac{A_{1}}{A} = \frac{2d}{l'},$$

$$= \frac{1}{10} \text{ if } l' + d = 20,$$

$$= \frac{1}{15} \text{ if } l' + d = 30,$$

$$= \frac{1}{20} \text{ if } l' + d = 40,$$

$$= \frac{1}{25} \text{ if } l' + d = 50.$$

Since about one-half of each diagonal bar is cut away to receive its end pin, we have for use,

$$\frac{A_1}{A} = \frac{4d}{l'},\tag{531}$$

$$= \frac{1}{5} \quad \text{if } l' + d = 20,$$

$$= \frac{1}{7.5} \quad \text{if } l' + d = 30,$$

$$= \frac{1}{10} \quad \text{if } l' + d = 40,$$

$$= \frac{1}{12.5} \quad \text{if } l' + d = 50,$$

$$= \frac{1}{15} \quad \text{if } l' + d = 60,$$

which is the ratio of the section of the diagonal bar to that of the two flanges, the section being normal to axis of the strut in both cases.

This ratio must be doubled for square struts latticed against deflection both ways, and it becomes

$$\frac{A_1}{A} = \frac{8d}{l'}, (532)$$

$$= \frac{1}{2.5} {if } l' + d = 20,$$

$$= \frac{1}{3.75} {if } l' + d = 30,$$

$$= \frac{1}{5} {if } l' + d = 40,$$

$$= \frac{1}{6.25} {if } l' + d = 50,$$

$$= \frac{1}{7.5} {if } l' + d = 60.$$

By reviewing our compression members, which are to be latticed in at least one direction, we find the girder diagonals having the ratio of length to radius of gyration = 100, giving ratio of length to diameter = about 40: so that, by (531), the weight found in (491) should be augmented by one-tenth of itself. Also, the vertical supports have a mean ratio of length to width =  $2 \times 10 = 20$ : so that, by (531), that part of their weight due to bending-moment, (521), should be augmented by one-fifth; or, which is approximately the same thing, the weight given in (522) is to be increased by one-tenth of itself. Similarly, we shall augment the weight of the lateral head struts, (526), by one-tenth of itself, on account of bracing.

In general, the longitudinal wind chords, being attached to the floor joists, to the transverse I-beams, and to the girder diagonals, will need diagonal bracing only when very long.

The top and bottom chords, however, though not having diagonal bracing in themselves, yet will need to have their weight, (484), (485), augmented by about one-tenth of itself, on account of the enlarged ends of I-bars, the re-enforcement of plates and rivets at joints, and the nuts and pins.

166. Weight of the Bridge. — Increasing, therefore, by one-tenth of itself, the weight of girders, of vertical supports, and of lateral head struts, and collecting all the weights which will then have been found in pounds, and expressed in terms of W, L, l, and h, for each value of n, the number of panels, and putting each sum = 2000nW = total weight of bridge, in pounds also, since W, the panel weight of bridge, and L, the panel weight of uniform discontinuous live load, are in tons, we find the following values of W for the different values of n, remembering that h and l are in feet:

(SEE FIG. 16.) PARABOLIC DOUBLE BOW, OR LENTICULAR GIRDER.

(0.001252l + 5.2647) + 0.000000000000000000000 + 0.00499l + 40.6914 + 64.15l - 1] $-1.146488/^{2} + 8000\% - 1.64657\%^{2}$ 43) + 0.13394978 + 178.59834] + 1.305595278+h[L(6.18225) + 24 + 24 + 42[L(W =

(533)

= 2.83403 tons, a minimum for h = 15.03893.if l = nL = 50,  $4.07998 + 1.616619h + 0.02116399h^{2}$  $-0.286622 + 0.8h - 0.000164657h^{2}$ []

n = 6

 $+ h^{2}[L(0.001307l + 10.16018) + 0.000000740l^{3} + 0.0177073l^{2} + 1.08926l + 66.05 + 176.91l^{-1}]$  $-1.5535727^2 + 12000h - 2.24059h^8$  $+h[L(6.2025f + 405) + 0.089592l^2 + 179.1833l] + 1.756856Ll^2$ 

W =

= 1.82101 tons, a minimum for h = 13.0178. 3.6601166 + 1.514252h + 0.02536247 $h^2$  if l = nL = 50, -0.388393 + 1.2h - 0.000224059 $h^2$ -0.388393 + 1.2hH

### % || | | |

 $+k[L(6.21262l + 567) + 0.067303^2 + 179.4758l] + 2.219138Ll^2 + 1.10410l + 104.628 + 344.09l + 1.10410l + 104.628 + 344.09l$  $-1.992705J^2 + 16000h - 2.83184h^2$ 

W =

= 393003 tons, a minimum for h = 24.1921.  $\frac{27.73922 + 3.347388h + 0.0596678h^2}{-1.992705 + 1.6h - 0.000232184h^2}$  if l = nL = 100, -1.992705 + 1.6h

## z = 10.

 $+h[L(6.2187l + 729) + 0.053895l^{2} + 179.651l] + 2.684697Ll^{2} + 1.11138l + 137.65 + 565.71l^{-1}]$  $-2443697l^{2} + 20000h - 3.41107h^{2}$ W =

= 3.12130 tons, a minimum for h = 22.6697.  $26.84697 + 3.201275h + 0.0660177h^3$  if l = nL = 100, - 0.000341107/t<sup>2</sup> -2.443697 + 2hU

n = 12.

 $+ h^{2}[L(0.0013394 + 32.58246) + 0.000000642/3 + 0.0160591/8 + 1.116194 + 170.66 + 841.75/-1]$  $-2.90056/^{2} + 24000h - 4.00759h^{2}$  $+k[L(6.22275J+891)+0.044943J+179.7683J]+3.151673LJ^{2}$ 11 Z

5.23077 tons, a minimum for h = 32.8472. 11 if l = nL = 150, 0.0002003795/12 + 0.05584925/13 44.3204 + 2.53908h -3.26313 + 1.2h -

11

n = 14

 $+ 0.000000627l^3 + 0.0156302l^2 + 1.11934l + 203.716 + 1172.18l - 1$  $-3.360842l^2 + 28000h - 4.61064h^2$  $+h[L(6.22564l + 1053) + 0.0385399l^2 + 179.8519l] + 3.619396Ll^2 + h^2[L(0.0013431l + 42.68822) + 0.000000627l^3 + 0.0156302l$ 

= 7.7226 tons, a minimum for h = 42.5850.  $\frac{103.41131 + 3.5171188h + 0.083867h^3}{-6.721684 + 1.4h - 0.000230532h^3}$  if l = nL = 200,

11

Z

n = 16

 $+ h^{2}[L(0.001345/ + 54.1126) + 0.000000615/^{3} + 0.0152773/^{2} + 1.12187/ + 236.74 + 1557.15/^{-1}]$  $-3.823293^{2} + 32000^{6} - 5.21739^{6}$  $(215) + 0.0337347^3 + 179.91457] + 4.08757127^3$ +h[L(6.227817 + 1

= 15.7507 tons, a minimum for h = 63.2967.  $344.8888 + 5741154\hbar + 0.1496117\hbar^{3}$  if l = nL = 300,  $-17.204818 + 1.6\hbar - 0.000260869\hbar^{3}$ 11

n = 18

 $+h^{2}[L(0.0013467 + 66.857718) + 0.0000000057^{3} + 0.01498437^{2} + 1.123897 + 269.77 + 1996.497^{-1}]$  $-4.2872445/^{2} + 36000h - 5.830715h^{2}$  $+h[L(6.22957 + 1377) + 0.0299947^2 + 179.9633^J] + 4.5560614L7^2$ 

= 47.96318 tons, a minimum for h = 110.4054. $= \frac{1581.9658 + 11.112554 + 0.3266634^{3}}{-53.59056 + 1.84 - 0.0002915364^{3}} \text{ if } l = nL = 500,$ -53.59056 + 1.84

$$n = 20$$
.

= 110.6006 tons, a minimum for h = 168.5605.  $+h^{2}[L(0.00134754 + 80.92249) + 0.00000059473 + 0.014738173 + 1.125767 + 302.80 + 2490.277 - 1]$  $-4.7522337^2 + 40000\% - 6.439829\%$  $+h[L(6.230854 + 1539) + 0.027l^2 + 180.002l] + 5.0247505Ll^2$  $4308.7235 + 17.28761h + 0.569255h^{2}$  if l = nL = 700,  $-116.42947 + 2h - 0.0003219915h^{2}$ W = M

# n = 22

 $+k^{2}[L(0.0013475J + 96.30706) + 0.0000005907l^{3} + 0.014529l^{2} + 1.12735J + 335.83 + 3038.47l^{-1}]$  $-5.21800847 + 44000h - 7.055422h^2$ +h[L(6.23195' + 1701) + 0.0245504'\* + 180.0342'] + 5.493587L'\*W =

= 151.3419 tons, a minimum for h = 202.445. $\frac{6392.5376}{-166.97627} + \frac{20.14436h}{2.2h} + \frac{0.719189h^2}{-0.000352771h^2} \text{ if } l = nL = 800,$ 

#### 24 || | | ≈

 $+h^{2}[L(0.001349' + 113.0124) + 0.000000584l^{3} + 0.0143494l^{2} + 1.12888l + 368.86 + 3641.11l^{-1}]$  $-5.684392l^{2} + 48000h - 7.67139h^{2}$  $+h[L(6.232877 + 1863) + 0.0225087^2 + 180.06087] + 5.962589L7^2$ W =

= 288.1332 tons, a minimum for h = 283.8575, $\frac{12422.0604 + 26.994836h + 1.0599919h^2}{-284.2196 + 2.4h - 0.0003835695h^2}$  if l = nL = 1000,

= infinity when l = 4h = 1947 feet, = infinity when l = 2h = 3157 feet, = 6h = 1356 feet, = 8k = 1034 feet, = infinity when i = infinity when ?

= infinity when l = 10h = 853 feet, = infinity when l = 12h = 697 feet, = infinity when l = 14h = 599 feet, = infinity when l = 16i = 525 feet,

> = 24: which are limiting spans for

In all these cases, the value of h which renders H a minimum has been found by the simple method of article 140, equations (469) and (470). The limiting spans just given have been determined by putting the denominator of (543) equal to zero, and substituting the assigned values of h. It will be seen that these limiting spans are independent of the live load, nL, and therefore represent the limit to the length of each girder imposed by its own weight. The effect of live load on the limiting span will be considered below.

167. Having found W and h, it is easy to compute the weights of all parts of the bridge from the expressions for weights in terms of W, h, L, and l. The computation affords a perfect verification of the accuracy of the work. We give below a table showing the number of panels and the height, which simultaneously render the total bridge weight, nW, a minimum for various spans ranging from 50 to 1,000 feet, and have thus probably extended the table far beyond any *economical* use of this girder.

Of course, we find great heights; but it should be remembered that one-half of this central height, h, is below the plane of the floor system, where the points of support are situated. Also the width, 18 feet; becomes too small for the highest girders; but it has been retained in this set of examples, to preserve uniformity in data.

To illustrate the change in central height and bridge weight, as the number of panels varies for the same span, we have given the solutions corresponding to three values of n, including that one which renders nW least. Also bridge weights and central heights are given for 2,000, 3,000, and 4,000 pounds of live load to the running-foot; the weights being minima values. The 30th line of this table exhibits the effect of a small live load upon the length of the limiting span, as resulting from the substitution of  $\frac{1}{5}IV$  for L in the equations for weight. Of course, we do not mean that the live load is small near the limit when W is infinite.

The reader cannot fail to notice how prolific in useful and interesting results these general equations for bridge weight are.

pa (9	Span Uniform lave load	-7	feet		33		<u> </u>	100			3 <u>5</u>	
****	Number of panels		feet feet	15.039 11.336	13 of 8 10 0,026 12 00, 8 1 8 24 0.219	6 11.471 10.932 6.1	24 192 31.440 31.440	30 27.570 31 213 37.37 10 4-41 0.312	18 31.285 31.285	10 34.300 63.008 1.5	32.847 32.847 62.759 77.57 12. 4.56 6.418	31.585 52.958 10\$
10	Raun of minimum dead to total load .	7+4			0.177			0.238			995	
	Weight of Parts, using Best Height and Two Thousand Founds for Knuming Foot of Span, as above, for Uniform Live Load		*									
- 2 2 7 1	Top chords, Equation (484) Do from head system, (528) Bottom chords, (485) Girder diagonals, (491) Flour, (402)		<u> </u>	1,893	1,271 1,271 1,271 1,667	1,374 1,374 1,076	8,783 137 5,653 4,414	9,141 104 5,872 4,821	9,524 162 6,112 5,729	22,098 558 14,197 10,930 26,000	72,734 556 14,588 11,922 26,000	\$3,43\$ 534 15,028 12,926 26,000
2 20	s, longitudinal, s, transverse, (907), (411, diagonals,		हैं कि हैं की	4,075 3,715 50t	4,215	4,526	1,00	9,657	5.433 10,057 1,816	14,670	12,725	15,724
28 = 2 5			<u> </u>	22,675	989	892 222 118 262 263	3,446 3,446 3,65 3,84 3,83 3,83 3,83 3,83 3,83 3,83 3,83	3,112 3,585 374 62,620	7,891 3,393 472 148 52,370	10,604 8,197 508 14,1,621	7,867 7,867 724 362 362	9,513 7,566 868 382 325,414
\$ 2 5		* 2	feet tons	16.853 19.45 19.45	14 51	12 79	27 18 39 37	25.07 39.38	23 62 39 35	33 841	36.14	34 54 80 17
2 82		# # ~	E Sage	18 10 16 70 648	15.92	15.72	#8.97 47.39	26.69 47.20 807	47.33	97.06	96.67	30-08 97-08 822
85	Limiting span when $W = \xi L$ , and $l = \xi h$ , Limiting span when $L = 0$ , and $l = \xi h$ ,	-	<u>5</u> 5	1,320	1,024	1,520		1,115	1,112			1,142

Two Double Parabolic-Bow or Brunel Girders. Highway Bridge, Fig. 16.

Continued.

- 0	Span Uniform live load	~ 7	feet		900			9008			00 00 00 00 00 00 00 00 00 00 00 00 00	
W+N0 F0 0 0	Panel length	R 本 美 本 本 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2	feet tons tons feet	12 43 960 108.279 163	14 42.585 108.117 137-04 147 4-70 0-351	14 41.47 108.503 13½	14 64.575 253.530 813	16 63.297 252.011 341.24 184 4.74 0.457	18 61.063 253.837 16\$	14 87.800 494.398 28‡	16 86.466 491.375 737.40 35 4.69 1.298	18 85,156 491.618
H 0 77 7 70 7 70 0 0	Weight of Parts, using Rest Height and I not Thousand Frands for Running-Foot of Man, as above, for Uniform Live Load.  Top chords, Equation (484) I've from head system, Equation (484) Button choices, system, Equation (484) Circles duage rale, (491) Flant, Includinal, (491) (491) Horizonta diagonals, (491) (503) Horizonta diagonals, (504) (503) Vertical amonoral		<b>#</b> # <b># # # # # # # # #</b>	43,769 1,325 21,300 34,667 34,667 2,667 2,669 2,669	28,593 24,667 34,667 34,667 31,186 31,118	45,810 1,171 29,365 34,667 16,300 21,739 2,711	119,228 4,289 76,470 52,000 41,914 13,736 12,749	120,189 4,024 77,427 57,1409 52,000 35,673 33,6673 12,3668 12,3668	122,792 3,741 78,686 61,865 52,000 33,600 33,600 12,902 12,742	25,874 10,369 161,168 95,364 69,333 745,14 43,632 188,050	254,528 9,770 103,154 104,020 60,333 67,832 48,832 83,778 83,778	177,396 164,941 172,713 69,333 57,936 46,007
38888888888 588888888888888888888888888	Head struts, (\$36)  Head diagonals, (\$34)  Total bradge weight (\$34)  Bridge weight per running-foot (\$11)  If live load = 3,000 lbs, per running-foot, then  If live load = 4,000 lbs, per running-foot, then  Limiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span when \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 10d, lamiting span \$L = 0, and \$L = 10d, lamiting span \$L = 10d, lamiting	****	Feren Frank	10,538 1,538 1,538 1,47,92 134,50 166,18	15,71 1,031	1365 1,265 1,065 1,065 1,146 1,146 1,146	1,776 2,1776 2,1776 1,776 1,084 70 87 341 38 75,22 368,50	204.02 1,680 1,680 321.42 331.43 389.33	323.45 1,585,674 1,585,674 1,585,674 323.31 71.36 828 1,147 1,147	9887.795 9887.795 9887.795 9611.50 102.04 748.09	284.750 24.457 24.457 29.735 74.657 74.657	983,236 24,53 24,53 924,50 97,54 97,54 97,54

TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS. HIGHWAY BRIDGE, FIG. 16.

Continued.

453,665 121,333 145,248 86,930 86,573 19055,111 224,326 8,293 8,293 22,616 2,216 630 49,072 849,283 177 95 2,697 48 185 64 3,108 28 6,333 44433,259 캶 18 20 169.875 168 560 2,214.140 2,212.012 159,740 87,737 01,881 1,047,975 1 226,223 7,417 23,784 17,000 2,632,48 187,79 3,142.15 \$2,274 844,841 420,117 4:153 3.160 1,311,400 1,318,703 4,424,025 6,320 œ Po 7,555 47 9 8 33 7 100,001 1 229,047 | 6,540 55.751 841.44 382.4 177.454 65.134 4,428,279 2,675 53 190 13 25,003 383 244.559 104.000 117,300 77.577 6,6 77 15.555 15.655 15.655 14,540 ,8:4,552 1,745 99 154 29 2,066 37 137,190 4,724 31,257 524,545 ዴ 8 138 470 145.94 122,040 152,040 152,040 152,040 153,040 153,040 153,040 833.544 813,582 33,389 521,380 272,087 5,331 2,36 4:33 \$ \$ 421.468 2,047.62 155,640 4,616 15,053 2,842,16 4,7,8 150,43 1,739 co 139.891 \$19.455 249,871 104,000 141,700 711,024 79,317 611,708 810,368 35.574 37 480,776 17,272 308,011 146,138 86,147 86,140 37,25,1 90,003 4,8-1 8,320 1,729,840 3.47. 117 711 1,081 74 123,71 1,294 8,0 8,0 1,47,1 109.160 864.930 5 8 110.405 863.316 490.35 27.9 3.453 119.44 1,076.30 1.25.84 476,345 18,495 305,245 181,703 80,067 \$55.55 \$5.58 \$1.40 14.10 1.727 8,714 3+0+4-18 97,420 1,720,024 33 3,45, 1,074,63 1,074,63 1,28,17 1,298.84 3,738 241,001 168,085 8,007 4. 1.5 372.042 IOO,227 57,793 9,242 tot, 375 16 113 258 864.808 100,05 \*1f feet tons feet Per Per tons PHO! het \* + \frac{1}{2} \f Worket of Paris, using Best Height and Two Flousand Founds for Running-boot of Span, as above, for Uniform Live Load \$200 P If live load = 3,000 lbs per running-foot, If live load = 4,000 lbs. per running-fixet, Lieft. I and ing span when L=0, and  $l=10\delta$ . Lamining span when W=5L, and  $l=5\hbar$ . Lamining span when L=0, and  $l=5\hbar$ . 33 93 33 3 (484) Equation (484) Ratio of minimum dead to total load Ratio of minimum dead to live load (<del>2</del>03) Panel length Ratto of length to height Lates builge weight. . . Broke weight if / = rost Top chords, Do from head system, Number of panels Best central height Undurm live load Bottom chords, <sup>\*</sup> Girdet dagonals, then. # NO F-00 9 2

HIGHWAY BRIDGE, FIG. 16. TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS.

Concluded.

H 6	Span	7"	feet		800			008			1,000	
W4N0 F0		24 m + 1	feet tons feet	205.026 3,333.240	20 203.670 3,326.093 43,189.70	22 202.445 3,329.522 36 H	244.242 4,86y.957 50	242.838 4,854.234 45	22 241.582 4,854 533 40 P	20 286.293 6,911.752 50	285.003 6,405 906 0	24 283.857 6,915.120 413
0 0	Ratio of length to height	7 1 1 1 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2			3.92			3.70			3.50	
9	Ratio of minimum dead to total load .	7+11			908.0			0.844		•	0.874	
	Weight of Parts, using Best Height and Two Thousand Pounds per Running-Foot of Span, as above, for Uniform Live Load											
11	Equation (					2,045,928	3,044,141	3,044,595	3,053,436	4,432,270	4,438,820	4,453,494
E # 1	i chords, diagonals, (		5	531,967	577,607	1,310,523		772,122	835,014	2,839,015	2,843,320	2,852,328
102	I-Beams, longitudinal, (503) (503) I-Beams, transverse, (507), (512)		bs.	231,822	208,639	130,007	293,400 115,568	264,060 118,438	240,054	326,000	296,364 138,453	271,667 271,667 142,003
8 6			. 108.	152,510	1,685,464	131,856		205,103	2,515,176	3,701,880	3,663,250	3,630,228
8 2 9	Vertical supports, (522) Head struts, (526) Head diagonals, (524)		25 E	332,153 7,893	328,100	325.233	9,403	10,685	11,959	12,597	•	631,200
24	Total bridge weight				6,652,186			9,708,463	9,709,067	13,823,505	13,8	3,830,240
200	If live load = 3,000 lbs. per running-foot, then.	**************************************		3,966.86	3,972.93	3,991.47	257.37	255.38	253.65 5,737.51	299.25	∞_	295.83 8,096.14
2 %	If live load = 4,000 lbs. per runnit	A = 2 = 2 = 2 = 2 = 2 = 2 = 2 = 2 = 2 =	feet	227.00	224.59	4,640.24	267.48	265.01 6,569.48	262.77	304.31	307.03	304.96
8 8 8	Limiting span when $L = 0$ , and $l = 104$ , Limiting span when $W = 5L$ , and $l = 5A$ ,	· · ·	मु हु हु हु									833 1,141 1,602
	;	•							1	<b>!</b>		

168. Among the inferences which may be legitimately drawn from the table of article 167 in regard to bridges having two lenticular Brunel girders of single system of the same width but of varying span, height, and uniform live load, are the following; viz. (see Fig. 116),—

1st, The best number of panels varies approximately as the cube root of the span,

$$n \propto l^{\frac{1}{3}}$$
 nearly. (544)

2d, The best central height for a uniform live load of 2,000 pounds per linear foot is about  $\frac{I}{4.2} \times \text{span}$ ,

$$h = \frac{1}{4.2} l \text{ nearly;} \tag{545}$$

and for different loads the best height for same span and same number of panels varies very nearly as the sixth root of the live load,

$$h \propto (nL)^{\frac{1}{6}}$$
 nearly. (546)

3d, For spans less than 500 feet, the least bridge weight varies approximately with the product of the best central height of girder multiplied by the span (that is, with the geometrical area of the girder, since the parabolic area is proportional to this product);

$$l < 500$$
,  $nW \propto lh$  nearly. (547)

4th, For the same span and same number of panels, and best central height of girder, the least bridge weight varies approximately with the square root of the uniform live load;

$$nW \propto (nL)^{\frac{1}{2}}$$
 nearly. (548)

Or, by reason of (546), the least bridge weight for same span

and best central height of girder varies nearly as the cube of this best central height;

$$nW \propto (nL)^{\frac{1}{2}} \propto h^3$$
 nearly. (549)

5th, For each span of 500 feet and over, a large increase of live load, and consequently of best height, causes a diminution in the number of panels corresponding to minimum bridge weight.

6th, Small deviations from the best height and best number of panels, or from either of them, do not greatly affect the bridge weight; but large deviations either way, in this respect, cause a great increase in bridge weight, as shown in sixth line of table,  $h = \frac{1}{10}l$ , thus rendering the girders only about one-half as high as minimum bridge weight requires.

7th, The limiting span increases slowly with the number of panels, till a maximum value depending upon  $\frac{W}{L}$  and  $\frac{l}{h}$  is reached.

8th, We cannot, for a given span, assign the best height and the best number of panels till we know the live load which is to be imposed.

169. Example. — Take span l = 200 feet, number of panels n = 14, central height of double parabola h = 42.585 feet, uniform live load nL = 200 tons = 400,000 pounds, width of  $2\frac{1}{2}$ -inch oak floor q = 16 feet, length of transverse I floor beams  $q_1 = 18$  feet. (See Fig. 16.) Loads applied at all apices equally by means of struts and suspenders which sustain the floor system in the plane of the axes of girders.

Assume that all cross-sections of members may be strictly adjusted to the developed strains.

 Load on 1 panel length of each longitudinal I-beam spaced 3.2 feet

$$=\frac{3.2}{16}\times\frac{434667}{14}=6209.53$$
 pounds.

By (502),

Cross-section of beam = 
$$S = 0.00015 \left( \frac{520}{3} \times \frac{200}{14} + 2000 \times \frac{200}{14} \right)$$
  
= 4.65714 square inches.

In order to satisfy the condition in (501), we must have, article 62,

$$I = \frac{1}{12}(bd^3 - b_1d_1^3) = \frac{l}{10n}Sd, \qquad (550)$$

$$S = bd - b_1d_1 = 4.65714;$$
 (551)

from which equations we find

$$O = (b^3 - bb_1^2)d^3 - 3b^2Sd^2 + \left(3bS^2 + 1.2\frac{l}{n}Sb_1^2\right)d - S^3. \quad (552)$$

Take b = 4.00 inches = breadth of flange.

 $b - b_1 = 0.26$  inch = thickness of web.

Then, from (552) and (551),

$$d = 9.080$$
 inches = depth of beam,

$$d - d_i = 0.614$$
 inches = depth of two flanges,

$$I = 60.466 = moment of inertia,$$

$$\frac{I}{d} = \frac{1}{8} \times \frac{12l}{n} \times \frac{6209.53}{2 \times 10000} = \frac{60.466}{9.08} = 6.65,$$

by reason of (502) and (52); the load being uniformly distributed on each panel length of beam, and these beams not being regarded continuous over the transverse beams.

Weight of these 6 longitudinal I-beams, by (503), equals

$$J_1 = 6 \times \frac{5}{18} \times 12 \times 200 \times 4.65714 = 18628$$
 pounds,

as given in preceding table.

It will be noticed that these beams are deep and comparatively thin; but, considering their area of cross-section, it will also be noticed that their moment of inertia is great as compared with ordinary beams of equal area of section.

Supported by the transverse I-beams, we have

Live load = 400000 pounds,

Floor = 34667 pounds,

Longitudinal I-beams = 18628 pounds.

Total for 14 panels = 453295 pounds.

Load on 1 beam = 32378 pounds.

From (504), we have

$$\frac{I}{d} = \frac{12 \times 18 \times 32378}{8 \times 2 \times 10000} = 43.7103 = 2S,$$
 by (505);

 $\therefore$  S = 21.855 square inches for vertical load.

But, in order to resist the assumed wind pressure,  $W_1 = 2,500\frac{h}{n} = 7,604$  pounds per panel, we must add to the cross-section due vertical load the areas found from (511), where now

$$Q_2 = \frac{8000}{1 + 0.93312 \times 0.07} = 7509$$
 pounds per square inch;

$$S_1 = \frac{2 \times 7604}{3 \times 14 \times 7509} (13^2, 12^2, 11^2, 10^2, 9^2, 8^2, 7^2);$$

$$= 8.149 \text{ square inches, 1st and 13th beams;}$$

$$= 6.944 \text{ square inches, 2d and 12th beams;}$$

$$= 5.835 \text{ square inches, 3d and 11th beams;}$$

$$= 4.822 \text{ square inches, 4th and 10th beams;}$$

$$= 3.906 \text{ square inches, 5th and 9th beams;}$$

$$= 3.086 \text{ square inches, 6th and 8th beams;}$$

= 2.363 square inches, 7th beam.

#### TOTAL SECTIONS.

• 
$$S + S_1 = 30.004$$
 square inches, 1st and 13th beams;  
28.799 square inches, 2d and 12th beams;  
27.690 square inches, 3d and 11th beams;  
26.677 square inches, 4th and 10th beams;  
25.761 square inches, 5th and 9th beams;  
24.941 square inches, 6th and 8th beams;  
24.218 square inches, 7th beam.  
 $351.962 = \Sigma S = \text{sum of all sections.}$ 

To satisfy the condition, (505), we must now have

$$I = \frac{1}{3}(bd^3 - b_1d_1^3) = 2Sd = 2d(bd - b_1d_1); \quad (553)$$

whence, eliminating  $d_{r}$ , we find

$$O = b(b^2 - b_1^2)d^3 - 3b^2Sd^2 + (3bS^2 + 24b_1^2S)d - S^3, \quad (554)$$

from which d may be found for each value of total section now called S;

$$d_i = \frac{bd - S}{b}.$$

Taking b = 5.5 inches = width of flange,  $b - b_1 = 0.7$  inch = thickness of web,  $b_1 = 4.8$  inches = difference, S = 30.0 square inches = cross-section, we find, by (554) and (553),

$$d=13.441$$
 inches = depth of beam,  
 $d_1=9.152$  inches = depth of web,  
 $d-d_1=4.289$  inches = depth of both flanges,  
 $\frac{1}{2}(d-d_1)=2.144$  inches = depth of one flange,  
 $I=806, \frac{I}{d}=60.$ 

Similarly may the proportions of the other transverse beams be found.

Or, if we choose to assume the thickness of web and of flanges, thus:

$$\begin{cases}
b - b_i = a \text{ (say)}, \\
d - d_i = c \text{ (say)},
\end{cases}$$
(555)

then we find, from (553),

$$O = d^3 - \frac{3}{2} \left( \frac{S}{a} + \epsilon \right) d^2 + \left\{ (12 + \frac{3}{2}\epsilon) \frac{S}{a} + \frac{c^2}{2} \right\} d - \frac{c^2 S}{2a}, \quad (556)$$

from which d is easily found either by trial or by Horner's Method.

Taking a = 0.7, c = 4, S = 30, we find, by (556),

$$d = 13.2, \quad \therefore d_1 = 9.2.$$

But 
$$b = \frac{S}{c} + a - \frac{ad}{c}$$
, by (553) and (555),  
= 5.889;  
 $\therefore b_{r} = 5.189$ ,

$$I = 792, \frac{I}{d} = 60.$$

Or again, by assigning values to d and  $d_r$  in (553), we find

$$b_1 = \frac{(24 - d)dS}{d_1(d^2 - d_1^2)},$$
 (557)

$$b = \frac{(24 - d)S}{d^2 - d_1^2} + \frac{S}{d}.$$
 (558)

Using two 12-inch beams for each panel point, we have

$$d = 12, d_1 = 10, d - d_1 = 2,$$

$$b = 5.3416$$
  $b_1 = 4.9098$   $b - b_1 = 0.4318$  inch.  
 $= 5.1272$   $= 4.7127$   $= 0.4145$  inch.  
 $= 4.9296$   $= 4.5311$   $= 0.3985$  inch.  
 $= 4.7490$   $= 4.3652$   $= 0.3838$  inch.  
 $= 4.5864$   $= 4.2156$   $= 0.3708$  inch.  
 $= 4.4404$   $= 4.0814$   $= 0.3590$  inch.  
 $= 4.3116$   $= 3.9629$   $= 0.3487$  inch.

Whatever be the form of beam section chosen, we hav Weight of 13 × 2 transverse I-beams

= 12 × 18 ×  $\frac{5}{18}$  \(\Sigma S = 60 × 351.962 = 21118 \) positions as per table.

Strains on the horizontal diagonals are given by where now  $W_1 = 7,604$ , and  $\sin \phi_1 = 0.78329$ ;

$$\therefore \frac{W_{\rm r}}{2n \times 15000 \sin \phi_{\rm r}} = 0.023114.$$

0.023114  $\times$  14  $\times$  13 = 4.206 square inches = section of 1st dia 0.023114  $\times$  13  $\times$  12 = 3.605 square inches = section of 2d dia 0.023114  $\times$  12  $\times$  11 = 3.051 square inches = section of 3d dia 0.023114  $\times$  11  $\times$  10 = 2.543 square inches = section of 4th dia 0.023114  $\times$  10  $\times$  9 = 2.080 square inches = section of 5th dia 0.023114  $\times$  9  $\times$  8 = 1.664 square inches = section of 6th dia 0.023114  $\times$  8  $\times$  7 = 1.295 square inches = section of 7th dia 18.444  $\times$  4 =  $\Sigma S$  = 73.776 square inch

Weight of 28 horizontal diagonals =  $12 \times \frac{5}{18} \times \frac{18}{\sin \phi_i} \times 73$ . = 5652 pounds,

as given in the table.

The cross-section of each panel length of a wind che shown in (514), thus:

$$\frac{lV_1l}{2nq_1Q} = \frac{7604 \times 200}{2 \times 14 \times 18 \times 6400} = 0.471478.$$

 $0.471478 \times 13 = 6.129$  square inches, 1st panel;  $0.471478 \times 24 = 11.316$  square inches, 2d panel;  $0.471478 \times 33 = 15.559$  square inches, 3d panel;  $0.471478 \times 40 = 18.859$  square inches, 4th panel;  $0.471478 \times 45 = 21.217$  square inches, 5th panel;  $0.471478 \times 48 = 22.631$  square inches, 6th panel;  $0.471478 \times 49 = 23.103$  square inches, 7th panel.

Total, 118.814 square inches for one-half of 1 gird

.. Weight of both wind chords = 
$$4 \times \frac{5}{18} \times \frac{12 \times 200}{14} \times 118.814$$
  
= 22631 pounds,

as by (515).

We now have, upon all vertical supports and abutments,

 $\frac{1}{28}$  × 481578 pounds = 17199 pounds,

 $+\frac{1}{26}$  × weight of transverse I-beams =  $\frac{21118}{26}$  pounds =  $\frac{812}{18011}$  pounds. Load on each vertical, article 163, =  $\epsilon_n$  =  $\frac{18011}{18011}$  pounds.

Therefore, by (516),

 $S = \frac{18011}{5333} = 3.3771$  square inches = cross-section of a strut, due vertical forces; and, by (518),

 $S_1 = \frac{18011}{6000} = 3.0018$  square inches = cross-section of a suspender, due vertical forces.

From (520),

 $S_2 = \frac{75}{170} \times \frac{200}{n} = 6.3025$  square inches = cross-section of each vertical, due bending-moment of assumed wind force;

...  $S + S_2 = 9.6796$  square inches for each strut,  $S_1 + S_2 = 9.3043$  square inches for each suspender. From (473), we have length of verticals,

	Suspenders.	Struts.	
$y = 0.43404 \times 13 =$	5.6425 feet.		r = 1
$0.43404 \times 24 =$		10.4170 feet.	2
$0.43404 \times 33 =$	14.3233 feet.		3
$0.43404 \times 40 =$	!	17.3616 feet.	4
$0.43404 \times 45 =$	19.5318 feet.		5
$0.43404 \times 48 =$		20.8340 feet.	6
$0.43404 \times 49 =$	21.2680 feet.		7
Sum required =	200.5268 feet.	194.4504 feet, for	all.

Longest strut, 20.834 feet = 250 inches;

therefore

Required radius of gyration =  $\frac{250}{100}$  =  $2\frac{1}{2}$  inches.

Each vertical may be made of 4 channels, 6 inches wide, each having an area of

2.4199 square inches for struts,

2.3261 square inches for suspenders,

latticed in pairs, and two pairs in one brace.

Weight of all vertical struts

 $= \frac{5}{18} \times 12 \times 194.4504 \times 9.679 = 6273 \text{ pounds,}$  Weight of all vertical suspenders

$$= \frac{5}{18} \times 12 \times 200.5268 \times 9.304 = \frac{6218 \text{ pounds.}}{12491 \text{ pounds.}}$$

Total,
Add one-tenth for lattice braces,

1249 pounds.

13740 pounds,

which accords with (522).

Equation (523) gives the cross-section of each head diagonal thus:

$$S = \frac{0.0208\frac{1}{3} \times 42.585 \times 200}{14 \times 18 \times 0.84609} = 0.831 \text{ square inch,}$$

which requires a round rod 1.056 inches in diameter if the ends are enlarged for cutting threads of screws.

Weight of the 8 head diagonals, by (524), is

$$8 \times \frac{5}{18} \times \frac{12 \times 2 \times 200}{14 \times 0.84609} \times 0.831 = 749$$
 pounds.

Cross-section of each head strut, by (525), is

 $\frac{1}{12} \times 42.585 = 3.549$  square inches. Add one-tenth for latticing, 0.355 square inch. 3.904 square inches.

Weight of 5 head struts, by (526) = 25 × 42.585 = 1065 pounds.

Add one-tenth for braces = 106 pounds.

Total,

Total,

Since for these head struts we have assumed

$$Q = 2500 = \frac{8000}{1 + \frac{x^2}{20000}}$$
 pounds per square inch,

$$\therefore x = 210 = \frac{q_1}{\rho} = \frac{12 \times 18}{\rho},$$

$$\therefore \rho = \frac{216}{210} = 1.03 \text{ inches} = \text{radius of gyration.}$$

We may therefore use, for each head strut, 2 4-inch channels latticed so that the web shall be 4 inches apart.

The increment of section of each top chord due to diagonal strain in head system is given by (527), thus:

$$S = 0.00271267 \times \frac{42.585 \times 200}{14} = 1.65$$
 square inches.

The total weight thus added along the 4 panel lengths of head system is

$$2 \times 4 \times \frac{5}{18} \times \frac{12 \times 2 \times 200}{14} \times 1.65 = 1257$$
 pounds, as by (528).

The strain in the top chords is given by equations (476), (481), and (482), where now

$$W + L = 7.72265 + 14.28571 = 22.00836$$
 tons.

For the segments of each top chord, the total strains due n(W + L) are

$$P_1 = 101.15 \text{ tons,}$$
  
 $P_2 = 96.55 \text{ tons,}$   
 $P_3 = 93.10 \text{ tons,}$   
 $P_4 = 91.55 \text{ tons.}$ 

Dividing these strains by the allowed inch strain, Q=3.2 tons, we get cross-sections,

$$S_1 = 31.6075$$
 square inches +  $S_2 = 30.1704$  square inches +  $S_3 = 29.0937$  square inches + 1.65 square inches,  $S_4 = 28.6078$  square inches + 1.65 square inches.

Now, the longest unsupported segment of top chord is

$$\frac{2l}{n\cos\alpha_2}$$
 = 29.405 feet = 352.86 inches,

.. 3.5286 inches = radius of gyration.

Therefore the top chord may be made up of 2 9-inch channels and a plate, or 2 plates 14 inches wide, and having such thickness as is required to complete the area of section.

Weight of top chords due load

$$= \frac{11}{10} \times \frac{5}{18} \times \frac{24l}{n} \sum_{\cos \alpha}^{S} = 44741 \text{ pounds,}$$

$$\frac{1257 \text{ pounds, due head system.}}{45998 \text{ pounds.}}$$
Total,

Similarly, for the segments of each bottom chord, the total strains due n(W + L) are, from equations (477), (481), and (483),

 $U_1 = 100.128 \text{ tons,}$   $U_2 = 99.513 \text{ tons,}$   $U_3 = 92.133 \text{ tons,}$  $U_4 = 91.374 \text{ tons.}$ 

These strains divided by 5, the allowed inch strain in tension, give the cross-sections of the successive segments of bottom chord in each girder,

 $S_1 = 20.0256$  square inches,  $S_2 = 18.9025$  square inches,  $S_3 = 18.4265$  square inches,  $S_4 = 18.2748$  square inches,

from which the links can easily be made up according to specified forms of body and head.

No change is here made on account of longitudinal component of lateral diagonal strain, since in the present case there is no lateral system between bottom chords, by reason of gravity.

Weight of all bottom chords increased by 10

$$= \frac{11}{10} \times \frac{5}{18} \times \frac{24l}{n} \sum_{\cos \beta} = 28695 \text{ pounds.}$$

The equations (490), (491), and (478) give cross-sections of alternate girder diagonals, thus:

 $S_1 = 5.115$  square inches,  $S_2 = 7.212$  square inches,  $S_3 = 8.664$  square inches,  $S_4 = 8.881$  square inches,  $S_5 = 7.750$  square inches,  $S_6 = 5.132$  square inches, for each of the two girders, the alternate set being the same inverted; and the weight of all is, calling  $Q_1 = \frac{8}{3}$ , and multiplying by 1.8  $\times$   $\frac{1}{10}$ , as specified,

$$4 \times \frac{5}{18} \times \frac{3 \times 1.8 \times 11}{8 \times 10} \times \frac{12 \times 200}{14} \Sigma \frac{S}{\cos \theta} = \frac{990}{7} \Sigma \frac{S}{\cos \theta}$$
$$= 23184 \text{ pounds.}$$

Now, since the longest unsupported length of any girder diagonal is

$$\frac{1}{2} \times \frac{l}{n \cos \theta_4} = 22.253 \text{ feet} = 267 \text{ inches,}$$

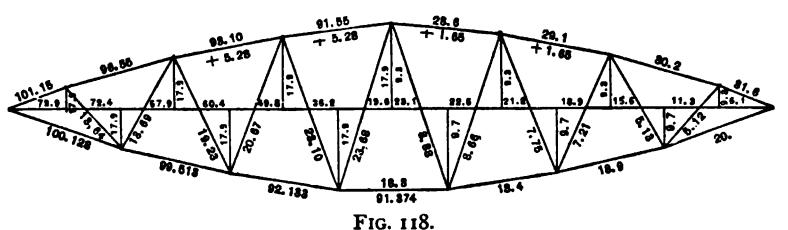
we have radius of gyration = 2.67 inches; and therefore 2 8-inch channels latticed 8 inches between webs will suffice for the longest diagonals.

We have thus determined the size and weight of all parts of this bridge, and find the total 216,233 pounds, as by table.

### STRAIN SHEET.

Strains, Tons: Cross-Sections, Square Inches.

For each of Two Girders.



Span, 200 seet; central height, 42.585 seet (best); uniform live load, 1 ton = 2,000 pounds per linear soot, applied at all apices by vertical members.

Regarding the greatest strains upon the chord pins as acting in "quadruple shear," and allowing 6,000 pounds as the inch strain in shearing, these pins will require a diameter of 3½ inches.

It remains to compute the deflection of this girder under the allowed chord strain of

$$B_1 = \frac{1}{2}(3.2 + 5) = 4.1$$
 tons per square inch.

For this purpose, use equations (318) and (319) combined thus:

$$D = \frac{B_1 a}{E h_1} \{ 1.386295 a - 2.302585 [(a + x) \log (a + x) + (a - x) \log (a - x) - 2a \log a ] \}, \quad (559)$$

where  $a = \frac{1}{2}l = 100$  feet = half-span, and x is measured from the centre. Also  $h_1 = h = 42.585$  feet, and we will call E = 24,000,000 pounds = 12,000 tons per square inch.

We have then, from (559),

Deflection 
$$D_1 = 1.3347$$
 inches for  $x = 0$ , centre;  
 $D_2 = 1.3150$  inches for  $x = \frac{100}{7}$ ;  
 $D_3 = 1.2549$  inches for  $x = \frac{200}{7}$ ;  
 $D_4 = 1.1521$  inches for  $x = \frac{300}{7}$ ;  
 $D_5 = 1.0006$  inches for  $x = \frac{400}{7}$ ;  
 $D_6 = 0.7911$  inch for  $x = \frac{500}{7}$ ;  
 $D_7 = 0.4954$  inch for  $x = \frac{600}{7}$ ;  
 $D_8 = 0$  inch for  $x = \frac{700}{7}$ , ends.

The proper camber may be given to the girders by equation (366), thus:

$$\lambda = \frac{3.2 + 5}{12000} \times \text{length of one parabolic chord.}$$

This length is given by equation (140); viz.,

$$S = \frac{1}{2}(200^{2} + 4 \times 42.585^{2})^{\frac{1}{2}} + 0.287823 \times \frac{200^{2}}{\frac{42.585}{2}} \log \frac{2h + (l^{2} + 4h^{2})^{\frac{1}{2}}}{200} = 205.5 \text{ feet,}$$

$$\lambda = \frac{8.2 \times 12}{12000} \times 205.5 = 1.685 \text{ inches}$$
= length to be added to top chord.

Or, each segment should be lengthened by

$$\frac{1.685}{7} = 0.241 = \frac{1}{4}$$
 inch nearly.

### SECTION 3.

### The Brunel Girder of Double System.

apply the dead and live loads at all apices by means of verticals whose upper half will act in tension, and lower half in compression. These verticals must also resist bending-moment due wind. Each girder has two equal parabolic chords, and the floor system is in the plane of girders' axes; each panel length of chord is straight, and the number of panels may be odd or even; each system will be assumed to do one-half of the work.

The height between the two parabolic arcs at the centre being h, the height at any apex is given by (474). Equation (473) gives y.

$$\tan \alpha_r = -\tan \beta_r = \frac{(y_r - y_{r-1})n}{l} = \frac{2h}{nl}(n - r - r_{-1}), \quad (560)$$

$$\frac{1}{\cos^2 \alpha_r} = \frac{1}{\cos^2 \beta_r} = 1 + \frac{4h^2}{n^2 l^2} (n - r - r_{-1})^2, \tag{561}$$

$$\tan \phi = -\tan \theta = (y_r + y_{r+1})\frac{n}{l}$$

$$= \frac{2h}{nl}[r(n-r) + r_{+1}(n-r_{+1})], \quad (562)$$

$$\frac{1}{\cos^2\phi} = \frac{1}{\cos^2\theta} = 1 + \frac{4h^2}{n^2l^2} [r(n-r) + r_{+1}(n-r_{+1})]^2.$$
 (563)

171. Moments at all apices due total dead and live uniform loads are given by (65),

$$M_r = \frac{W + L}{2n} l(n-r)r;$$

and the horizontal component of chord strain is

$$H_r = \frac{M_r}{h_r} = (W + L)\frac{nl}{8h};$$
 (564)

that is, this component is uniform throughout the girder under uniform load.

Strain in top chords 
$$= P = \frac{H}{\cos \alpha},$$

Strain in bottom chords 
$$= U = \frac{H}{\cos \beta},$$

Cross-section of top chords = P + Q, Q = 3.7647;

Cross-section of bottom chords = U + T, T = 5.0000.

Volume of a segment of top chord is, therefore,

$$\frac{12lH}{nQ\cos^2\alpha} = \frac{3}{2}(W+L)\frac{l^2}{Qh\cos^2\alpha} \text{ cubic inches.}$$
 (565)

Weight of top chords, in pounds,

$$= \frac{3}{2} \frac{m(W+L)l^2}{Qh} \sum \sec^2 \alpha$$

$$= \frac{3 \times 5(W+L)l^2}{2 \times 18Qh} \left\{ n + \frac{4h^2}{3l^2} \left( n - \frac{1}{n} \right) \right\}$$

$$= \frac{W+L}{h} \left\{ 0.1106771nl^2 + 0.14757 \left( n - \frac{1}{n} \right) h^2 \right\}$$

$$= \frac{W+L}{h}$$

$$0.442709l^{2} + 0.55338h^{2}$$

$$0.553386l^{2} + 0.70833h^{2}$$

$$0.664063l^{2} + 0.86082h^{2}$$

$$0.774740l^{2} + 1.01190h^{2}$$

$$0.885418l^{2} + 1.16211h^{2}$$

$$0.996095l^{2} + 1.31172h^{2}$$

$$1.106772l^{2} + 1.46093h^{2}$$

$$1.217449l^{2} + 1.60984h^{2}$$

$$11.328126l^{2} + 1.75853h^{2}$$

$$12.438804l^{2} + 1.90705h^{2}$$

$$13.549481l^{2} + 2.05542h^{2}$$

$$14.660158l^{2} + 2.20370h^{2}$$

$$15.770835l^{2} + 2.35188h^{2}$$

$$16.881512l^{2} + 2.50000h^{2}$$

$$17.992190l^{2} + 2.64804h^{2}$$

$$18.2102867l^{2} + 2.79610h^{2}$$

$$19.2213544l^{2} + 2.94400h^{2}$$

$$20.324221l^{2} + 3.09192h^{2}$$

$$21.434898l^{2} + 3.23981h^{2}$$

$$22.545576l^{2} + 3.38767h^{2}$$

$$23.2656253l^{2} + 3.53554h^{2}$$

$$24.3656253l^{2} + 3.53554h^{2}$$

### Similarly, we have

Weight of bottom chords, in pounds,

$$= \frac{W+L}{h}$$

$$0.916667l^{2} + 1.21212h^{2}$$

$$1.000000l^{2} + 1.32407h^{2}$$

$$1.083333l^{2} + 1.43590h^{2}$$

$$1.166667l^{2} + 1.54762h^{2}$$

$$1.250000l^{2} + 1.65926h^{2}$$

$$1.333333l^{2} + 1.77083h^{2}$$

$$1.416667l^{2} + 1.88235h^{2}$$

$$1.500000l^{2} + 1.99383h^{2}$$

$$1.583333l^{2} + 2.10526h^{2}$$

$$1.666667l^{2} + 2.21667h^{2}$$

$$1.750000l^{2} + 2.32804h^{2}$$

$$1.833333l^{2} + 2.43939h^{2}$$

$$1.916667l^{2} + 2.55072h^{2}$$

$$2.000000l^{2} + 2.66204h^{2}$$
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172. For the advancing uniform live load of  $\frac{1}{2}L$  at each upper and lower apex, or of L at each vertical section through apices, we have at foremost end, by (64) and (474),

$$(H_L)_r = \frac{M_r}{h_r} = \frac{\frac{Ll}{2n^2}r(r+1)(n-r)}{\frac{4h}{n^2}r(n-r)} = \frac{Ll}{8h}(r+1), \quad (568)$$

and at one interval before the foremost end of live load, by (68) and (474),

$$(H_L)_{r+1} = \frac{M_{r+1}}{h_{r+1}} = \frac{\frac{Ll}{2n^2}r(r+1)(n-r-1)}{\frac{4^{h}}{n^2}(r+1)(n-r-1)} = \frac{Ll}{8h}r. \quad (569)$$

Therefore

$$\Delta H = (H_L)_{r+1} - (H_L)_r = -\frac{Ll}{8h},$$
 (570)

which is the horizontal component of strain on both diagonals of a panel, on the present assumption that the two diagonals do equal work, and that the whole load is on one girder.

Hence, for each of two girders, we shall have

Cross-section of a girder diagonal = 
$$\frac{\frac{1}{4}Ll \times 1.8}{8hQ_1 \cos \theta}$$
, (571)

according to our specifications for members alternately in compression and tension. Putting  $m = \frac{5}{18}$ ,  $Q_1 = \frac{8}{3}$ , we find

Weight of girder diagonals, pounds, =  $\frac{4 \times 12ml \times 0.45Ll}{8nhQ_1} \sum \sec^2 \theta$ 

$$= \frac{0.28125Ll^2}{h} \left\{ 1 - \frac{2}{n} + \frac{4h^2}{15l^2} \left( 2n^2 - 5 - \frac{30}{n} + \frac{63}{n^2} - \frac{30}{n^3} \right) \right\}$$
(572)  

$$= \frac{L}{h} \left\{ 0.28125 \left( 1 - \frac{2}{n} \right) l^2 + 0.075 \left( 2n^2 - 5 - \frac{30}{n} + \frac{63}{n^2} - \frac{30}{n^3} \right) h^2 \right\}$$
  

$$= \frac{L}{h} \left\{ 0.140625 l^2 + 1.72266 h^2 \right\}$$

$-\frac{2}{n}$ ) $l^2 + 0.075 \left(2n^2 - 5 - \frac{3}{n}\right)$	$\frac{0}{n}+\frac{03}{n^2}-\frac{30}{n^3}\Big)h^2\Big\}$
	n
$0.140625l^2 + 1.72266h^2$	4
$0.168750l^2 + 3.09600h^2$	5
$0.187500l^2 + 4.77083h^2$	6
$0.200892l^2 + 6.74344h^2$	7
$0.210937l^2 + 9.01318h^2$	8
$0.218750l^2 + 11.58025h^2$	9
$0.225000l^2 + 14.44500h^2$	10
$0.230113l^2 + 17.60781h^2$	11
$0.234375/^2 + 21.06901h^2$	12
$0.237980l^2 + 24.82886h^2$	13
$0.241071l^2 + 28.88759h^2$	14
$0.243750l^2 + 33.24533h^2$	15
$0.246093l^2 + 37.90228h^2$	16
$0.248161l^2 + 42.85854h^2$	
	17
$0.250000l^2 + 48.11420l^2$	18
$0.251644l^2 + 53.66934h^2$	19
$0.253125l^2 + 59.52403h^2$	20
$0.254463l^2 + 65.67833h^2$	21
$0.255682l^2 + 72.13228h^2$	22
$0.256803l^2 + 78.88592h^2$	23
$0.257812l^2 + 85.93179h^2$	24

All girder diagonals must be so constructed as to transmit stresses of tension or compression.

173. Collecting the weights now found for top and bottom chords and girder diagonals, we find

Weight of girders due to loads, pounds,

$$\begin{array}{l} = \frac{W}{h} \quad 0.776042l^2 + 0.97005h^2 \\ 0.970053l^2 + 1.24166h^2 \\ 1.164063l^2 + 1.50897h^2 \\ 1.358073l^2 + 1.77381h^2 \\ 1.552085l^2 + 2.03711h^2 \\ 1.746095l^2 + 2.56093h^2 \\ 2.134116l^2 + 2.82196h^2 \\ 2.522137l^2 + 3.08260h^2 \\ 2.716148l^2 + 3.60304h^2 \\ 2.910158l^2 + 3.86296h^2 \\ 3.104168l^2 + 4.12271h^2 \\ 3.298179l^2 + 4.64187h^2 \\ 3.686200l^2 + 4.90136h^2 \\ 4.074221l^2 + 5.41996h^2 \\ 4.268231l^2 + 5.93839h^2 \\ 4.656253l^2 + 6.19758h^2 \\ \end{array} \begin{array}{l} 0.916667l^2 + 2.69271h^2 \\ h \\ \hline \\ 0.916667l^2 + 2.69271h^2 \\ h \\ \hline \\ 1.138803l^2 + 4.33766h^2 \\ 1.351563l^2 + 6.27980h^2 \\ 1.351563l^2 + 6.27980h^2 \\ 1.351563l^2 + 8.51725h^2 \\ 1.763022l^2 + 11.05029h^2 \\ 1.964845l^2 + 13.87962h^2 \\ 2.165105l^2 + 17.00593h^2 \\ 10.2364229l^2 + 20.42977h^2 \\ 2.165105l^2 + 17.00593h^2 \\ 2.562501l^2 + 24.15161h^2 \\ 12.2562501l^2 + 24.15161h^2 \\ 12.2562501l^2 + 24.15161h^2 \\ 12.366202l^2 + 3.86296h^2 \\ 3.153908l^2 + 37.10829h^2 \\ 3.546340l^2 + 47.24089h^2 \\ 3.742190l^2 + 52.75607h^2 \\ 18.3937844l^2 + 58.57070h^2 \\ 19.328684l^2 + 71.09829h^2 \\ 4.133336l^2 + 64.68470h^2 \\ 20.328684l^2 + 71.09829h^2 \\ 21.32937h^2 + 22.3937h^2 \\ 22.3913l^2 + 77.81148h^2 \\ 22.3913l^2 + 77.81148h^2 \\ 22.3913l^2 + 77.81148h^2 \\ 22.3913l^2 + 77.81148h^2 \\ 22.3913l^2 + 92.12937h^2 \\ 24.914065l^2 + 92.12937h^2 \\ 24.914$$

This weight of girders is to be increased by one-tenth of itself, as explained in article 165. Also, it will be augmented to meet the strain brought upon the top chords by the head system.

174. Make the floor of  $2\frac{1}{2}$ -inch oak planks, 52 pounds per cubic foot, and of the width of q feet. Then, if q = 16 feet,

Weight of floor = 
$$\frac{2.5}{12} \times 52ql = \frac{520}{3}l$$
 pounds. (573)

175. Longitudinal I Floor Beams; Conditions and Weights given in Article 157. — We may further explain the assumption in (501) thus: Taking an analytical table of ordinary wrought-iron I-beams, we may easily see, that, for depths of 8 inches and upwards, we have approximately

$$\frac{r^2}{d^2} = 0.15, (574)$$

r being the radius of gyration, and d the depth of beam. Now, by equation (184),

 $r^2=\frac{I}{S};$ 

hence if we take, as we manifestly may,  $d = \frac{2}{3} \frac{l}{n}$ , and eliminate r, we shall find

$$\frac{I}{d} = 0.15Sd = \frac{l}{10n}S,$$

as in (501), where the same notation is used, d being in inches, and l in feet.

We obtain, from (503),

Weight of 6 wrought-iron longitudinal I-beams, in pounds,

$$=6Ll+l^2\frac{0.5^2}{n}=J_1 \tag{575}$$

176. Also, let the transverse I-beams be conditioned as in article (158); then (507) yields, taking S from (506),

Weight of (n-1) transverse I-beams due load, pounds, =  $(n-1) \times \frac{5}{18} \times 12 \times 18S$  (576)

1	j	!			n
= 1	$5.2650 + l^2$	0.003949 + Ll	0.18225 + L	243	4
	5.6160	0.003369	0.19440	324	5
	5.8500	0.002925	0.20250	405	6
	6.0171	0.002579	0.20828	486	7
	6.1425	0.002303	0.21262	567	8
	6.2400	0.002080	0.21600	648	9
	6.3180	0.001895	0.21870	729	10
	6.3817	0.001740	0.22091	810	11
	6.4350	0.001609	0.22275	891	12
	6.4800	0.001495	0.22431	972	13
	6.5186	0.001397	0.22564	1053	14
	6.5520	0.001304	0.22680	1134	15
	6.5812	0.001234	0.22781	1215	16
	6.6071	0.001166	0.22871	1296	17
	6.6300	0.001105	0.22950	1377	18
	6.6505	0.001050	0.23021	1458	19
	6.6690	0.001000	0.23085	1539	20
	6.6857	0.000955	0.23143	1620	21
	6.7009	0.000914	0.23195	1701	22
	6.7148	0.000876	0.23244	1782	23
	6.7275	0.000841	0.23287	1863	24

177. Equation (512) becomes, when n is odd,

Weight of iron to be added to transverse I-beams on account of wind, in pounds,

$$= h \left(\frac{25}{24} + 0.972 \frac{n}{l}\right) \left(7n - 12 + \frac{5}{n}\right) (577)$$

178. When n is odd, we use  $\frac{n-1}{2}$  terms in summing (509), adding  $4 \times \frac{1}{8}(n^2-1) = 2\left(n-\frac{n+1}{2}\right)\left(n-\frac{n-1}{2}\right)$  for the two diagonals of the middle panel, and find, as in (509), Weight of horizontal diagonals, in pounds,

$$= 4 \times \frac{12q_1}{\sin^2 \phi_1} \times m \times \frac{W_1}{2nT_1} \begin{cases} n(n-1) + (n-1)(n-2) \\ + (n-2)(n-3) \\ + (n-3)(n-4) \end{cases} \\ = \frac{3.5}{6}h \left(n - \frac{1}{n}\right) \left(1 + \frac{l^2}{324n^2}\right) \end{cases}$$

$$= h \begin{vmatrix} 28.000 + hl^2 \\ 40.000 \\ 51.852 \\ 63.636 \\ 75.385 \\ 87.111 \\ 98.823 \\ 110.526 \\ 122.222 \\ 133.913 \end{vmatrix} \begin{vmatrix} n(n-1) + (n-1)(n-2) \\ + (n-2)(n-3) \\ + (n-3)(n-4) \\ 0.0034568 \\ 5 \\ 0.0034568 \\ 5 \\ 0.0034568 \\ 0.0016232 \\ 11 \\ 0.001949 \\ 15 \\ 0.0010554 \\ 17 \\ 0.0009450 \\ 19 \\ 0.0008554 \\ 21 \\ 0.0007813 \end{vmatrix}$$

179. In summing (515) for odd values of n, we use  $\frac{n-1}{2}$  terms of the series, and add  $\frac{1}{8}(n^2-1)$  for middle panel, then multiply the sum by 4, since the two wind chords are to be alike.

Weight of wind chords, in pounds,

180. We shall now have, at the centre of each vertical, the load, e<sub>n</sub>, as defined in article 163. Therefore

Cross-section of lower half of vertical due  $\varepsilon_{\pi}$ , in square inches,

$$=S=\frac{\frac{1}{2}\epsilon_{n}}{Q_{3}}=\frac{\epsilon_{n}}{10667};$$
 (580)

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Weight of all lower halves of vertical due  $e_n$ , in pounds,

$$= 2 \times \frac{5}{18} \times \frac{\epsilon_n}{10667} \times 12 \Sigma y = 0.000208 \frac{1}{3} \left(n - \frac{1}{n}\right) h \epsilon_n. \quad (581)$$

Cross-section of upper half of vertical due  $\varepsilon_n$ , in square inches,

$$= S_{1} = \frac{\frac{1}{2}e_{n}}{T_{1}} = \frac{e_{n}}{12000}; \qquad (582)$$

Weight of all upper halves of verticals due  $\varepsilon_n$ , in pounds,

$$= 2 \times \frac{5}{18} \times \frac{\epsilon_n}{12000} \times 12\Sigma y = 0.000185 \left(n - \frac{1}{n}\right) h \epsilon_n. \quad (583)$$

As in the second part of article 163, so here, suiting the expression to the changed length of chord segments, we have, from the assumed wind pressure, the moment

$$62.5\frac{l}{n}y = \frac{1}{2}S_2 \times \frac{1}{10}yB_1, B_1 = 5667.$$

**Therefore** 

Cross-section of any vertical due bending-moment, in square inches,

$$= S_2 = \frac{15l}{68n}; (584)$$

Weight of verticals required to resist bending-moment due wind, in pounds,

$$= 4 \times \frac{5}{18} \times \frac{15l}{68n} \times 12\sum y = 0.9803922 \left(1 - \frac{1}{n^2}\right) hl. \quad (585)$$

Adding together the three expressions, (581), (583), and (585), the sum is

Weight of all verticals, pounds,

$$= h \left\{ 0.000393518 \left( n - \frac{1}{n} \right) \epsilon_n + 0.9803922 \left( 1 - \frac{1}{n^2} \right) l \right\}$$
 (586)

```
= h | 0.001152Ll + 1.5425L + 0.000000744l^3 + 0.00002495l^2 + 0.95631l + 0.0032
     0.001179Ll + 1.9654L + 0.000000699l^3 + 0.00002002l^2 + 0.97747l + 0.0055
     0.001194Ll + 2.3885L + 0.000000677l^3 + 0.00001725l^2 + 0.99043l + 0.0081
     0.001203Ll + 2.8077L + 0.000000650l^3 + 0.00001490l^2 + 0.99821l + 0.0115
     0.001209Ll + 3.2245L + 0.000000633l^3 + 0.00001310l^2 + 1.00383l + 0.0152
     0.001213Zl + 3.6396Z + 0.000000616l^3 + 0.00001168l^2 + 1.00770l + 0.0197
     0.001216Ll + 4.0536L + 0.000000606l^3 + 0.00001054l^2 + 1.01060l + 0.0245
     0.001218Ll + 4.4668L + 0.000000593l^3 + 0.00000960l^2 + 1.01289l + 0.0300
                                                                                 II
     0.001220Ll + 4.8793L + 0.000000585l^3 + 0.00000881l^2 + 1.01475l + 0.0359
                                                                                 12
     0.001221Ll + 5.2914L + 0.000000577l^3 + 0.00000814l^2 + 1.01637l + 0.0425
                                                                                 13
     0.001222Ll + 5.7031L + 0.000000570l^3 + 0.00000756l^2 + 1.01767l + 0.0494
                                                                                 14
     0.001223Ll + 6.1145L + 0.000000564l^3 + 0.00000706l^2 + 1.01885l + 0.0571
                                                                                 15
     0.001224Ll + 6.5257L + 0.000000559l^3 + 0.00000663l^2 + 1.01992l + 0.0651
                                                                                 16
     0.001224Ll + 6.9367L + 0.000000555l^3 + 0.00000623l^2 + 1.02090l + 0.0739 | 17
```

```
= h \begin{vmatrix} 0.001225Ll + 7.3474L + 0.000000549l^3 + 0.00000590l^2 + 1.02178l + 0.0829 \\ 0.001225Ll + 7.5811L + 0.000000546l^3 + 0.00000553l^2 + 1.02263l + 0.0928 \\ 0.001225Ll + 8.1686L + 0.000000543l^3 + 0.00000531l^2 + 1.02342l + 0.1029 \\ 0.001226Ll + 8.5797L + 0.000000539l^3 + 0.00000505l^2 + 1.02418l + 0.1138 \\ 0.001226Ll + 8.9894L + 0.000000537l^3 + 0.00000483l^2 + 1.02491l + 0.1250 \\ 0.001226Ll + 9.3997L + 0.000000533l^3 + 0.00000483l^2 + 1.02560l + 0.1370 \\ 0.001226Ll + 9.8092L + 0.000000532l^3 + 0.00000443l^2 + 1.02627l + 0.1492 \\ 24
```

since for the even values of n we have  $\varepsilon_n$ , given in article 163, and for the odd values

```
\epsilon_{5} = 0.62430Ll + 1040.5L + 0.010821l^{2} + 18.0353l + 0.001850hl^{2} + 5.925h + 14.58\frac{h}{l},

\epsilon_{7} = 0.44593Ll + 1040.5L + 0.005521l^{2} + 12.8823l + 0.001204hl^{2} + 6.131h + 21.38\frac{h}{l},

\epsilon_{9} = 0.34683Ll + 1040.5L + 0.003340l^{2} + 10.0196l + 0.000881hl^{2} + 6.237h + 28.19\frac{h}{l},

\epsilon_{11} = 0.28378Ll + 1040.5L + 0.002236l^{2} + 8.1978l + 0.000691hl^{2} + 6.302h + 34.99\frac{h}{l},

\epsilon_{13} = 0.24012Ll + 1040.5L + 0.001600l^{2} + 6.9366l + 0.000567hl^{2} + 6.345h + 41.80\frac{h}{l},

\epsilon_{15} = 0.20810Ll + 1040.5L + 0.001202l^{2} + 6.0117l + 0.000480hl^{2} + 6.376h + 48.60\frac{h}{l},

\epsilon_{17} = 0.18362Ll + 1040.5L + 0.000935l^{2} + 5.3045l + 0.000416hl^{2} + 6.399h + 55.40\frac{h}{l},

\epsilon_{19} = 0.16429Ll + 1040.5L + 0.000742l^{2} + 4.7461l + 0.000366hl^{2} + 6.417h + 62.21\frac{h}{l},

\epsilon_{21} = 0.14864Ll + 1040.5L + 0.000613l^{2} + 4.2940l + 0.000327hl^{2} + 6.432h + 69.01\frac{h}{l},

\epsilon_{23} = 0.13571Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},

\epsilon_{23} = 0.13571Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},

\epsilon_{19} = 0.16429Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},

\epsilon_{24} = 0.14864Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},

\epsilon_{19} = 0.14864Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},

\epsilon_{24} = 0.14864Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},

\epsilon_{10} = 0.14864Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},

\epsilon_{10} = 0.14864Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},

\epsilon_{10} = 0.14864Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},

\epsilon_{11} = 0.14864Ll + 1040.5L + 0.000511l^{2} + 3.9205l + 0.000295hl^{2} + 6.444h + 75.82\frac{h}{l},
```

Here, as in article 163, we have put  $\frac{1}{6}l$  for h in the last three terms of the value of  $\epsilon_n$ ; a substitution introducing no practical error in the small resulting terms, but enabling us to keep our final equation down to the second degree with respect to h.

181. For the head lateral system, we proceed as in article 164, now having a pair of diagonals and a strut for each panel so far as the head system extends, say to n-6 of the central panels when head room is sufficient.

Hence, by (480), the moment at  $\left(\frac{n-1}{2}\right)^{th}$  panel point is, since  $W_1 = 2.500\frac{h}{n}$ ,

$$M = \frac{\frac{1}{4}W_{1}}{2n}l\left(n - \frac{n-1}{2}\right)\frac{n-1}{2} = \frac{W_{1}l}{3^{2}}\frac{n^{2}-1}{n},$$

$$= 78.125\frac{n^{2}-1}{n^{2}}hl(n \text{ odd});$$

and at the  $\left(\frac{n}{2}\right)^{\text{th}}$  panel point,

$$M = \frac{\frac{1}{4}W_1}{2n}l\left(n - \frac{n}{2}\right)\frac{n}{2} = \frac{W_1 ln}{3^2},$$
  
= 78.125hl (n even);

$$H = \frac{M}{q_1} = 78.125 \frac{n^2 - 1}{n^2} \frac{hl}{q_1}$$
 (n odd),

= 
$$78.125 \frac{hl}{q_1}$$
 (*n* even);

$$\Delta H = H \times \frac{2}{n} = 156.25 \frac{n^2 - 1}{n^3} \frac{hl}{q_1}$$
 (n odd),

$$= 156.25 \frac{hl}{nq_1} \qquad (n \text{ even});$$

requiring each diagonal tie to resist

$$\frac{156.25}{\cos \alpha \cos \phi_1} \times \frac{hl}{q_1} \times \frac{n^2 - 1}{n^3} \text{ pounds } (n \text{ odd}),$$

or

$$\frac{156.25}{\cos\alpha\cos\phi_1}\times\frac{hl}{nq_1}$$
 pounds (n even),

and to have a cross-section

$$S = \frac{156.25hl(n^2 - 1)}{15000\cos\alpha\cos\phi_1 n^3 q_1},$$

$$= \frac{0.0104\frac{1}{8}(n^2 - 1)hl}{n^3 q_1\cos\phi_1} \text{ square inches } (n \text{ odd}), \quad (587)$$

or

$$S = \frac{0.0104 \frac{1}{6}hl}{nq_1 \cos \phi_1} \text{ square inches } (n \text{ even}); \qquad (588)$$

calling, as before,  $\cos \alpha = 1$ .

Length of each head diagonal 
$$=\frac{l}{n\cos\alpha\cos\phi_1}=\frac{l}{n\cos\phi_1}$$
 practically.

Weight of 2 (n - 6) wrought-iron head diagonals, in pounds,

$$= 2(n-6) \times \frac{5}{18} \times \frac{12l}{n\cos\phi_1} \times \frac{0.0104\frac{1}{6}(n^2-1)hl}{n^3q_1\cos\phi_1}$$

$$= 0.003858 \frac{(n^2-1)(n-6)}{n^4\cos^2\phi_1}hl^2,$$

$$= 0.003858(n^2-1)(n-6)\left(\frac{hl^2}{n^4} + \frac{324h}{n^2}\right)(n \text{ odd}), \quad (589)$$

$$= 0.003858(n-6)\left(\frac{hl^2}{n^2} + 324h\right) \qquad (n \text{ even}), \quad (590)$$

since 
$$q_1 = 18$$
 feet, and  $\frac{1}{\cos^2 \phi_1} = 1 + \left(\frac{18n}{l}\right)^2 = 1 + \tan^2 \phi_1$ .

Multiplying the cross-section, (587), (588), of head diagonal by  $2 \times 10,000 \cos \phi_1 \tan \phi_2$ , we find, after dividing by 2,500 pounds inch strain,

$$S = \frac{0.0104\frac{1}{6} \times 2 \times 10000(n^2 - 1)hl \tan \phi_1}{2500n^3q_1},$$

$$= \frac{1}{12} \frac{n^2 - 1}{n^2} h \text{ square inches } (n \text{ odd}), \qquad (591)$$

$$= \frac{0.0104\frac{1}{6} \times 2 \times 10000hl \tan \phi_1}{2500nq_1},$$

$$= \frac{1}{12}h \text{ square inches } (n \text{ even}), \qquad (592)$$

$$= \text{cross-section of head strut.}$$

Weight of (n - 5) head struts, in pounds,

$$= (n-5) \times \frac{5}{18} \times 12 \times 18 \frac{(n^2-1)}{n^2} \frac{h}{12}, \quad (593)$$

$$= 5 \frac{(n-5)(n^2-1)}{n^2} h.$$
 (n odd),

$$= \frac{5}{18}(n-5)\frac{h}{12} \times 12 \times 18, \tag{594}$$

$$= 5(n-5)h \qquad (n \text{ even}),$$

Calling the compression along each segment of top chord due head diagonals equal to

$$\Delta H = 156.25 \frac{n^2 - 1}{n^3} \frac{hl}{q_1}$$
 or  $156.25 \frac{hl}{nq_1}$ 

according as n is odd or even, and taking the allowed inch strain in compression, as above, viz.,

$$Q = \frac{8000}{1 + \frac{50^2}{40000}} = 7529 \text{ pounds} = 3.764 \text{ tons,}$$

we have

Cross-section of iron to be added to segments of top chord in head system, square inches,

$$= \frac{156.25}{7529} \cdot \frac{n^2 - 1}{n^3} \cdot \frac{hl}{18}$$
 (n odd),

$$= S = 0.00115295 \frac{n^2 - 1}{n^3} hl, \qquad (595)$$

$$= \frac{156.25}{7529} \frac{hl}{18n}$$
 (n even),

$$= 0.00115295 \frac{hl}{n}. \tag{596}$$

Weight of added iron in (n - 6) panels for top chords, pounds,

$$= 2(n-6) \times \frac{5}{18} \times \frac{12l}{n} S,$$
 (597)

$$= 0.0076863 \frac{(n-6)(n^2-1)}{n^4} h l^2 \qquad (n \text{ odd}),$$

$$= 0.0076863 \frac{n-6}{n^2} h l^2 \qquad (n \text{ even}),$$

1		n			n
$= hl^2$	_	4	$= hl^2$	<b>0.00</b> 030609	15
	_	5		0.00030024	16
	0	6		0.00029155	17
	0.00015366	7		0.00028468	18
	0.00024020	8		0.00027603	19
	0.00028116	9		0.00026902	20
	0.00030745	10		0.00026085	2 I
	0.00031427	II		0.00025409	22
	0.00032026	12		0.00024654	23
	0.00031648	13		0.00024020	24
	0.00031373	14			

182. As explained in article 165, we shall here augment, by one-tenth of tself, each of the following expressions just found; viz.,—

The girders proper,

The vertical supports, and

The lateral head struts.

Then, adding together all the parts of the complete bridge, and putting the sum = 2,000n W, the weight of any bridge in pounds, we derive the following values of W, in terms of L, l, and h, for the different values of n.

Then, by assigning values to L and l, differentiating, and putting  $\frac{dW}{dh} = 0$ , we get W a minimum, and h best, as in article 140, equations (469) and (470).

(899)

-7

 $+ 1.051947 + 40.692 + 64.157^{-1}$  $+ h^{2}[L(0.0012677 + 4.65868) + 0.0000008187^{3} + 0.02019167^{3}]$  $-0.8536462l^2 + 8000h - 1.067055h^2$  $h[L(6.18225J + 243) + 0.133949J^2 + 178.5983J] + 1.0083332J^2$  $\mathcal{U} =$ 

= 2.7156 tons, a minimum for h = 13.4222 feet.  $\frac{1}{2} = \frac{1}{2} 9h + 0.02041786h^2$ 3.1510417 + 1.61661-0.2134115 + 0.8h11

*n* == 5.

 $+ 1.07522l + 53.006 + 116.65l^{-1}$  $h[L(6.1944 + 324) + 0.107369/^{3} + 178.9493/] + 1.252683Ll^{2} + h^{2}[L(0.001297/ + 6.9333) + 0.000000769/^{3} + 0.0185239/^{2}]$  $-1.0670587^2 + 100004 - 1.365834^2$ 

 $\mathcal{U} = \mathcal{U}$ 

= 2.1368 tons, a minimum for h = 12.742 feet.  $3.1317075 + 1.5553087h + 0.02254876h^{\circ}$  if l = nL = 50, h - 0.000136583/12 -0.2667645 + 11

n = 6.

 $h[L(6.2025f + 405) + 0.0895917l^2 + 179.1833f] + 1.486719LJ^2 + 1.08947l + 66.05 + 176.91l^{-1}] + h^2[L(0.001313f + 9.53508) + 0.000000745f^3 + 0.0177074l^3 + 1.08947l + 66.05 + 176.91l^{-1}]$ 

*W* =

= 1.7676 tons, a minimum for h = 12.072 feet. if l = nL = 50,  $= \frac{3.09733125 + 1.51425h + 0.02484294h^2}{-0.320117 + 1.2h - 0.000165987h^2}$ 

 $-1.280469/^{2} + 12000/h - 1.65987/h^{3}$ 

n = 7.

 $+ h^{2}[L(0.00132334 + 12.45747) + 0.000000715/3 + 0.0168778/9 + 1.09803/ + 79.2976 + 256.62/-1]$  $+ 0.0768647l^3 + 179.3504l] + 1.714861Ll^8$ h[L(6.208281 + 486)]W =

= 1.5111 tons, a minimum for h = 11.504 feet.  $-1.493887^2 + 14000h - 1.95119h^9$  $= 3.062252 + 1.484835h + 0.02710699h^2 \text{ if } l = nL = 50, \\ -0.37347 + 1.4h - 0.000195119h^3 \text{ if } l = nL = 50, \\$ 

# n = 8 (without head system).

$$N = \frac{h[L(6.21262l + 567) + 0.067303^{2} + 179.4758l] + 1.939324Ll^{3}}{+ 0.000000696l^{3} + 0.016503l^{2} + 1.10421l + 92.378 + 344.09l^{-1}}, (602)$$

$$\frac{3.030194 + 1.462724^h + 0.02943693h^2}{-0.426823 + 1.6h - 0.000224082h^2}$$
 if  $l = nL = 50$ ,  $l = 1.3351$  tons, a minimum for  $h = 10.996$  feet.

11

$$n = 8$$
 (with head system).

$$h[L(6.21262l + 567) + 0.067303l^{2} + 179.4758l] + 1.939324Ll^{2} + h^{2}[L(0.001330l + 15.7022) + 0.000000696l^{3} + 0.0167341l^{2} + 1.10421l + 111.378 + 344.09l^{-1}], (603) -1.707293l^{2} + 16000h - 2.24082h^{2}$$

$$= \frac{24.24155 + 3.347388h + 0.059122h^2}{-1.707293 + 1.6h - 0.000224082h^2}$$
 if  $l = nL = 100$ , = 3.7885 tons, a minimum for  $h = 22.629$  feet.

 $+ h^{2}[L(0.001334l + 19.27118) + 0.000000678l^{3} + 0.0163029l^{3} + 1.10847l + 131.0088 + 451.01l^{-1}]$  $-1.920704l^{2} + 18000l - 2.52931h^{2}$  $h[L(6.216l + 648) + 0.0598578l^2 + 179.5733l] + 2.161329Ll^2$ 11 Ź

$$= \frac{24.01477 + 3.2662h + 0.062568h^2}{-1.920704 + 1.8h - 0.000252931h^3}$$
 if  $l = nL = 100$ , = 3.3586 tons, a minimum for  $h = 21.918$  feet.

$$n = 10$$
.

 $33767 + 23.16548) + 0.0000000677^3 + 0.016007667^2 + 1.111667 + 151.152 + 565.717^{-1}$  $-2.134116/^{2} + 20000/ - 2.81702/^{3}$  $+ 0.0538957^2 + 179.6513^7] + 2.381616Ll^2$  $+ h^{2}[L(0.001)]$ h[L(6.2187l + 729)]11 Z

$$= \frac{23.81616 + 3.201278h + 0.06617111h^2}{-2.134116 + 2h - 0.000281702h^2}$$
 if  $l = nL = 100$ , = 3.0248 tons, a minimum for  $h = 21.252$  feet.

(000)

n = 11.

 $+ h^{2}[L(0.00133987 + 27.38623) + 0.0000006527^{3} + 0.01569337^{2} + 1.114187 + 170.777 + 699.854^{-1}]$  $-2.3475287^2 + 22000h - 3.10416h^3$  $h[L(6.22091l + 810) + 0.0490127l^2 + 179.715f] + 2.600652Ll^2$ 11 Z

= 5.4878 tons, a minimum for h = 31.575 feet.  $7.9792732 + 0.51830h + 0.010740469h^{8}$  if l = nL = 150,  $-0.5281938 + 0.22h - 0.0000310416h^{3}$ 11

n = 12.

 $+ k^2 [L(0.001342l + 31.93397) + 0.000000643l^3 + 0.015465l^2 + 1.11622l + 190.9345 + 841.75l^{-1}]$  $-2.560939^{2} + 24000h - 3.39086h^{2}$  $h[L(6.22275J + 891) + 0.044942J^2 + 179.7683J] + 2.818751LI^8$ 11

X

5.0347 tons, a minimum for h = 30.917 feet. 11  $-0.5762113 + 0.24h - 0.0000339086h^2$  if l = nL = 150,  $7.927737 + 0.507816h + 0.01115805h^2$ H

n = 13

 $h[L(6.22431I + 972) + 0.041495I^2 + 179.8123I] + 3.036129LI^2$ 

(808)  $+ h^{2}[L(0.0013437 + 36.80949) + 0.0000000357^{3} + 0.01522887^{8} + 1.118017 + 210.5615 + 1003.117^{-1}]$  $-2.774351l^2 + 26000h - 3.67724h^8$ W =

= 7.9452 tons, a minimum for h = 40.773 feet;  $18.6838 + 0.7172783h + 0.01623834h^{2}$  if l = nL = 200,  $-1.10974 + 0.26h - 0.0000367724h^{2}$ 11

nW = 126.072 tons if 10h = l = nL = 200.

n = 14

 $+ h^{2}[L(0.001344^{2} + 42.01309) + 0.000000027l^{3} + 0.0150484l^{4} + 1.11944^{2} + 230.7163 + 1172.18l^{-1}]$  $h[L(6.22564^7 + 1053) + 0.03853997^2 + 179.8519^7] + 3.252941Ll^8$ W =

= 16.9912 tons, a minimum for h = 60.957 feet.  $= \frac{62.73529 + 1.200104h + 0.0285066h^2}{-2.0889867 + 0.28h - 0.0000396334h^3}$  if l = nL = 300,

 $-2.9877637^3 + 28000h - 3.963344h^3$ 

, (611)

, (612)

# \* = 15.

(610)  $+ k^2 [L(0.0013453' + 47.545) + 0.000000052'^3 + 0.01486539'^3 + 1.12073' + 250.351 + 1360.8' -1]$ -3 201174° + 30000k - 4.24936k²  $A[L(6.22680l + 1134) + 0.035971l^{2} + 179.8853l] + 3.469299Ll^{3}$ M =

= 15.85998 tons, a minimum for h = 60.320 feet.  $\frac{62.44738^{2} + 1.1724378^{4} + 0.029047043^{6}}{-2.8810566 + 0.3^{6} - 0.0000424926^{6}}$  if I = nL = 300, Ħ

# # = 16.

 $+ h^{2}[L(0.0013464/ + 53.40576) + 0.00000015/3 + 0.0147202/3 + 1.12191/ + 270.4936 + 1557.15/-1]$ -3.414585/3 + 320004 - 4.5349843A[L(6.277817 + 1215) + 0.0337347 + 179.9145/] + 3.685287L/8|| |<u>|</u>

= 28.6796 tons, a minimum for k = 81.339 feet. $-5.46334 + 0.32k -0.0000453498k^2$  if l = kL = 400, 147.4115 + 1.70016344 + 0.044663564

H

# n = 17.

k[L(6.228717 + 1296) + 0.03175427\* + 179.94047] + 3.900974L7\* + 1.79.94047] + 3.900974L7\* + 3.9009

=46.9617 tons, a minimum for k=104.126 feet.  $286.83632 + 2.2762507k + 0.06347744k^2$  if l = nL = 500,  $-9.07 \pm 0.34 h - 0.0000482058 h^{3}$ 1

# # 118.

 $+ h^{2}[L(0.00134754 + 66.11378] + 0.0000000047^{3} + 0.01445647^{3} + 1.1247 + 310.272 + 1996.97^{-1}]$ -3.841409/2 + 360004 - 5.106057 AP  $A[L(6.2295f + 1377) + 0.029094^{2} + 179.9633f] + 4.116409Ll^{3}$ 11

= 44.4024 tons, a minimum for h = 103.511 feet.  $\frac{285.86174 + 2.22254 + 0.06421043^{13}}{-9.6035^2 + 0.364 - 0.00005106057^{13}} \text{ if } l = nL = 500,$ II

 $+ h^{2}[L(0.00134754 + 72.767) + 0.000000001/3 + 0.0143378/2 + 1.12489/4 + 329.9347 + 2239.49/-1]$  $-4.05482l^2 + 38000h - 5.391496h^2$  $h[L(6.230217 + 1458) + 0.0284137^2 + 179.98387] + 4.331628L7^3$  $W = \cdot$ 

= 105.7218 tons, a minimum for h = 157.380 feet.  $781.9728 + 3.5430075^{l} + 0.11067843^{l}$  if l = nL = 700,  $-19.86862 + 0.38^{l} - 0.00005391496^{l}$ 

11

n = 20.

 $+ h^{2}[L(0.0013475/ + 80.1386) + 0.000000597l^{3} + 0.0142392l^{2} + 1.12576l + 350.05 + 2490.27l^{-1}]$  $-4.2682327^8 + 40000h - 5.676737h^3$  $h[L(6.230854 + 1539) + 0.0277^2 + 180.0023J] + 4.54667LI^2$ 

W =

= 149.9221 tons, a minimum for k = 188.687 feet;  $-27.31668 + 0.4h - 0.000056767674h^{3}$  if l = nL = 800, nW = 124.521 tons if l = 10h = nL = 200.  $1163.9475 + 4.22229h + 0.13921166h^2$ 11

n = 21.

 $+ h^2[L(0.0013486l + 87.6458) + 0.0000005929l^3 + 0.0141409l^2 + 1.126598l + 369.728 + 2760.48l^{-1}]$  $-4.481643^2 + 42000h - 5.961956k^8$  $h[L(6.23143/ + 1620) + 0.0257169/^2 + 180.019/] + 4.761552L/^3$ M = M

= 206.7529 tons, a minimum for h = 223.620 feet; -  $36.301309 + 0.42h - 0.00005961956h^{2}$  if l = nL = 900, 1652.93877 + 4.9263154 + 0.17081376/2

nW = 124.630 tons if l = 10h = nL = 200.

n = 22

 $h[L(6.23195^l + 1701) + 0.0245504l^2 + 180.0342l] + 4.976304Ll^2 + 1.1274l + 335.8275 + 3038.47l^{-1}]$  $-4.695054^{2} + 44000h - 6.24712h^{3}$ W =

= 283.4025 tons, a minimum for h = 261.545 feet;  $\frac{2261.95636 + 5.651732h + 0.2098764h^{3}}{-46.95054 + 0.44h - 0.00000624712h^{3}}$  if l = nL = 1000, nW = 124.350 tons if l = 10h = nL = 200. 11

= 3516 feet, 6h = 1506 feet, 8k = 1148 feet, 4h = 2163 feet, 925 feet, = infinity when l = 10b = 100= infinity when l = = infinity when ! = W = infinity when I == infinity when ! = In equation (619),

94 feet, 12h = 774 feet,  $14^{l} = 665$  feet, 16h = 583 feet, infinity when l = 100h = 100= infinity when ! = infinity when ! = infinity when ! = 11 11 11 K

1200

when the number of panels is  $24 = \pi$ , and h and l are related as above. These limiting values the denominator of (619) equal to zero, as usual. which are limiting spans of I are found by putting

Double Height of Girder and Number of Panels yielding Minimum Bridge Weight, Web System. Limiting Span. HIGHWAY BRIDGE, FIG. 22. Two Double Parabolic-Bow or Brunel Girders. Uniform Live and Dead Loads applied at all Appen.

١

Span   Span
See
George   G
See   12.04   1.506   1.506
Sect   12,000   1,00
(\$566)
(\$50) (\$573) (\$7
(\$56) (\$573) (\$73)
(\$56) (\$573) (\$73)
udunal, best udunal, serse, (576) (onals, (593) (515) (515) (515) (516)
ight panel panel is a see of the
一。 艾罗斯 医 原 医多头 化 沙摩夫 "是我,是 " 第二 "我们就是一个一个一
the state of the s
Number of panels, best central height Least bridge weight Too chords, Too chords, Gorder diagonals, Floor, I-Beams, transverse, Horroutal diagonals, Wind chords, Vertical supports, Head struts, Head struts, Head struts, Ratio of length to cel Ratio of length to cel Ratio of length to cel Ratio of length to cel Ratio of length to cel Ratio of length to cel Ratio of length to cel Ratio of length to cel Ratio of length to cel Ratio of minimum d load Least bridge weight Best Least bridge weight Least bridge weight Least bridge weight Limiting span, I = 10 Limiting span, I = 10
# @: = = = = = = = = = = = = = = = = = =

Uniform Live and Dead Loads applied at all Apices. Height of Girder and Number of Panels yielding Minimum Bridge Weight. Double Weiform Live and Dead Loads System. Limiting Span, — Continued. TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS. HIGHWAY BRIDGE, FIG. 22.

10 to 00   Number of panels, best   18   18   14   18   18   18   18   18	1 44	1+70	Span	~	ğ		\$00			988			400	
Dec. for bread breight   Dec.   14.437   14.43	\$ I.o	\$ too	•	*	-	128	13	14	13	14	15	114	1.5	16
10   Teart bridge weight   Fig. 69   10   10   10   10   10   10   10   1	3 10		Best central height		Ĭ	41,437	40.788	40.147	689,19	60-957	60,330	82.637	81.990	81 339
10	_		1	212	tons:	103 304	103.288	103.450	\$38.085	237 877	937.900	459.480	458.782	458.874
Context dagonals,   (577)	_		9		ď,	37,67x	38,206	38,777	200,930	080*101	102,057	214,056	215,408	216,981
Bottom chords,   (573)	_		_		8	531	510	द्ध	1,737	1,721	1,662	4,148	4,015	3,907
Context diagonals	_		_		Ä,	±8,364	28,767	261.66	75,994	76,785	77,516	161,178	162,182	163,374
1.0			•		Ž,	£51'0¢	21,088	666'18	47,694	90'8'6 <del>4</del>	52,119	89,695	63,900	196,092
10   10   10   10   10   10   10   10			(573)		ġ.	34,667	74,667	34,567	52,000	5x,000	\$2,000	69,333	69,333	69,333
10	_		(\$45)		Š,	21,733	20,002	18,629	45,138	41,914	39,120	74,515	69,546	65,200
Vertical supports,   (579)   158,   5,370   5,370   12,374   12,374   11,370   11,370   12,374   11,370   11,370   12,374   11,370   11,370   12,374   11,370   12,374   11,370   12,374   11,310   11,310   11,			(225)		<u>8</u>	20,233	90,577	20,000	31,370	31,804	32,202	43,153	43-738	44,283
Head struits,   5559    158,   22,343   21,015   21,355   71,575   72,575	_		(22)		8.	2,300	5,72	5,338	12,294	11,009	11,742	23,064	22,017	122,143
10 "Head diagonals, (\$90), (\$90)   150.   15			•		<u> </u>	22,343	21,010	21,330	74,219	72,991	71,077	175.075	173,202	171,000
1.0 " Head diagonals, (589), (590) hs. 2555					\$ 2	43,573	13,105	1000	00000	30,273	100.00	50.514	25,00	55,490
10 Total least bridge weight to central height, \$\frac{4}{4} \times \text{ fort }			-		<u> </u>	40000	11/0	3.2	000	3,010	31303	0000	4,400	9.0
10 Bridge weight per running foot	Ξ		à			306 606	900 can	400 900	4190	6/44	200	0000	6,4,3,5	277
1.0 "Ratio of length to central height, \( \frac{7}{4} \) fort \( \frac{1}{4} \) fort \( \f			Bridge weight per running foot .		1	1,033	1,072	1,016	1.083	1.584	1.086	2.207	2.204	2.204
10 Ratio of length to central height, \$\( \triangle + \triangle + \triangle + \triangle + \triangl			Panel length	4 17	į		100		1			284	4.90	1
10 "Raito of mitimum dead to live W±L as to 5164 a 5173 a 0.7936 a 0.7939 a 1.1487 a 0.3466 a 3405 a 0.3409 a 0.4439 a 0.4439 a 1.1487 a 0.3466 a 3405 a 0.4439 a 0.4	_		Ratio of Jenoth to central height.	14		4.827	4.006	4.082	4.867	4.021	4.073	840	A. Bro	4.018
load Rate of minimum dead to total $\frac{W+L}{W+L}$ 0.3465 0.5164 0.3773 0.7926 0.7929 0.7929 1.1487 1.0 If least of minimum dead to total $\frac{W+L}{W+L}$ 0.3465 0.3405 0.3409 0.4423 0.4423 0.4423 0.4423 0.4423 0.4423 0.5346 1.5 Best simult $\frac{R}{A}$ feet 45.467 44.914 68.585 07.926 0.4423 0.58.739 1.5 Least bridge weight lancous, $\frac{R}{A}$ feet 45.747 1.31.826 1.25 $\frac{R}{A}$ 1.0 1.0	_		Ratio of minimum dead to live	-			-				-		e) and	
1.0   Matto of minimum dead to total   11				TTM		0.5165	P0.5164	0.5173	0.7936	0.7929	0.7930	1.1487	1.1469	11.472
1.0 "			_	4		,						`		
1.0 " Iff   16   17   18   19   30   30   30   30   30   30   30   3				17.7		0.3400	9-34DS	0,3400	0.4425	0.4489	0.4423	0.5340	0.5343	0.5343
1.0 ** Best sumul- ** feet 46.46 45.77 44.91 301.900 299.398 302.713 695.739 1.5 ** Least number of panels simul- ** feet 46.46 45.77 44.914 68.585 67.004 90.681 1.5 ** Least number of panels simul- ** feet 49.734 48.685 47.725 73.430 71.396 77.196 97.589 2.0 ** Best central height lancous, ** first 1.0 ** 1.5 ** feet 49.734 48.685 47.725 73.430 71.396 97.589 2.0 ** Least bridge weight lancous, ** first 1.481 1.481 1.481 1.479.72 1.779.69			f panels,	nt.		18	18	2	**	#2 88	7	OF.	60	ä
1.5 " Best simul- # feet 46.646 45.747 44.014 68.585 67.026 67.014 90.681 1.5 " Least number of panels simul- # feet 46.646 45.747 131.826 303.579 302.873 302.887 302	_			MIN	_	124.483	124.436	124.531	301.000	200-30g	304.713	695.739	623.3zī	626.786
15 Least			-1000	R-		7	18	20	a i	130	14	14	16	91
Least Least	-		taneous.		_	40.040	45-747	44.914	68.585	67.920	67.014	90 081	89-273	88.877
2.0 " Best central height laneous, will tons 159.786 159.768 159.796 356.588 366.081 366.384 695.465 2.0 Linuting span, $l = xok$ , $l$ feet 924.88 1.481 1.482 1.492 1.479.69 1.779.69 1.779.69 1.779.69 1.779.69 1.779.69 1.779.69 1.779.69 1.779.69	<u> </u>	4		¥ ;	_	132.035	131.777	131,826	505 573	302.873	302 830	578.076	578-844	579,002
2.0 "Least bridge weight lancous, will tone 159,786 159,768 159,796 366,538 366,338 695,465  L=0 Limiting span, l = 10k			f simul-	R -4		10 10	48 68r	200	4 4 20	900	ar ion	of a re	1 40	20 000
L=0 Limiting span, l= rok , leet 924.88 1,481 1,499 1,477 1,479.64 1,779.64 1,779.72 1,779.69 1,779.64	_		t (laneous,	1		150,780	10,568	250.700	200.030	100	266.35B	605.465	100	600,344
$L = 0$ Limiting span, $l = \cos \frac{l}{l}$ ,				ĸ		15	16	17	18	19	8	7.1.7		
L=2W Limiting span, l = 54 l feet 1,479.81 1,779.70 1,779.69 1,779.67 1,779.67 1,779.67 1,779.67	7	°-	Limiting span, $l = rok$ .		_	924.88								
L = 0 Limiting spain, t = 56	м.	*	Limiting span, / = 54		ĕ,	I,483	1,481	1,479	1477	1,474	1,470			
	<u>٦.</u>	0	<b>3</b> ,		ž	1,779.83	1,779.70	1,779.72	1,779.09	1,779.07	1,779.04			

Double Unaform Live and Dead Loads applied at all Apices. Height of Carder and Number of Panels yielding Manumum Bridge Weight.

Web System, Limiting Span, — Continued. TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS. HIGHWAY BRIDGE, FIG. 22.

	ļ												
7# =	1+7u	Spatt	~	ğ		000			90		_	100	
0,	1.0 ton	Number of panels, best	*		16	1.7	18	18	17	~	65	19	8
_	*	Best central height		를	104 738	gzı tor	103 511	130 503	129 897	124 186	т.	157.380	156 787
9.		weight	N F	totte	798 445	798.349	790.244	1,298 631	7,24,7 507		2,008 775	2,008 714	2,012 428
5,0		Top chords, Eq. (566)		28	399,309	401,380	403,790	677,700	680,105		$\vdash$	1,095,746	1,100,806
0.1	2	system,		<u>8</u>	7,B63	7,589	7,367	14,305	13,613		_	21,286	20,068
7 1.0	*	Pottom chords, (567)		lòs,	300,656	302,215	304,030	510,268	512,078	_	_	815,030	828,900
8 1.0	:	diagonals, (		Š	156,654	163,657	170,627	332,040	242,8 18		358,219	374,054	384,762
0.1		~		8	86,667	86,667	80,067	200,401	104,000	_		121,333	121,333
10, 1.0	: :	~~		Š.	101,875	688'56	90,556	146,700	138,070	130,400		168,145	159,740
				ď.	56,377	\$7,083	57.746	060'69	69,870	70,013		85,206	86,158
	: :	(925) (605) (926)		Š.	39,140	37,704	35,559	688170	05120	00,024	_	20,200	47,324
13 1,0				ŏ.	344,052	340,353	337,028	D17,307	011,410	4 11,000	-	1,000,350	993.388
14, 1.0		oorts,		ď.	92,701	92,029	91,339	144,517	143,654	142,581		470,01E	\$10,02b
13.		Head struts, (593), (594)		Ŋ,	6,337	61849	7,400	568,₹	8,543	9,244	_	12,095	12,035
16 1.0		Head diagonals, (58y), (590)		e S	5-255	5,236	5,750	11/18	8,623	0,50	_	13,235	13,118
0'1 Zr		Total least bridge weight		8	1,596,891	1,596,698	1,598,490	2,597,262	2,595,014	5,596,179	+	4,017,429	4,024,857
18 1.0		Bridge weight per running-foot .		4	3,693	3,193	3,197	4+343	4,325	4,327	_	5,739	5,749
19 1.0		Panel length		feet	314	30 -	67.0	372	35 7	333		3614	35
		th to central	9+7		+-774	4.803	OE 9	4-59B	4.619	1704	4-433	4.448	4.465
21 I.o		Ratio of minimum dead to live										1	
_	7		14		1.5969	1.5967	1.5985	9,1644	E.1695	g.re35	a.8697	3,3696	2.8749
0'1			¥		-			0.69.0	A 6810	0.8810		y	411
			7+41		60.01	0.0249	6,0153	oronoo	af an in	of on the	019250	0.7410	61420
_	= 3	If/= ( panels	*										
24 1.0			* 5	tone	-	1	(	1		1	Ť	*	;
-		Best   Best   Limit-	2-	3	-	10	91	GT.	97	900 000	•	01	2.7
	=	f and	é,	100	00000	750	1 14.003	200 a 200 a	141 200	200		2 422 813	201 727
200					14		16	14	10	14.33		10	16
	=			ğ	122.000	121.Box	110.520	150.417	140,146	140 115	182.504	181 141	180.081
		Least Lancous,	MK	toms	1,184 732	1,184.657	1,189.604	1,882 195	1,880.760	1,882.939	2,845.514	2,842,088	2,844.075
21 / 10	0	Limiting area / - tok	R ~	3									
, ,	1 100	I demitting appear	. ^	3									
3 2	1	Lamining span,		ğ 3									
) )	}	a disposa Burning	•	2									
-	1		1	1		1			-!    -		-	1	

Double Uniform Live and Dead Loads applied at all Apices Height of Girder and Number of Panels yielding Minimum Bridge Weight, Weight, — Concluded. HIGHWAY BRIDGE, FIG. 22. TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS.

P	1+7m	Span	~	ž	_	800			006			1000	
Çi	1.e fon	Number of panels, best	*		18	19	98	18	19	98	18		02
_	: Q	Best central height	-	ğ	189-793	189,265	188.687	ğ	224.687	2	264 381	263.B74	263-332
9,1		•	A:W	tons					4,333.528	933	6,127,960	_	6,134 S22
1.0		Š		PB.			1,684.587		2,487,316	_	3,587,317	P-1	3,597,617
		(265)		<u>. 6</u>			32,489		50,236	6 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	75,264		70,841
2 1.0	•	(292)		Š.	1,261,053		1,268,394	1,870,459	1,872,798	386,878,1	2,701,038	**	2,708,793
-	: :	,		<b>z</b>	487,550	500,875	531,959	645,403	075,590	705,048	435,150		914,971
_				ź.	138,007	138,007	136,007	150,000	150,000	150,000	173,333	173,333	173,333
				ď.	231,322	219,017	300,000	0071F60	277-954	304,000	308,233		320,000
_		Louise transverse, (570), (577)		1	507.00	99.572	980,001	113,300	114,054	115,944	120,020	130,475	121,000
2 6	ج و و			į	4411179	133,303	1,501,470	9.376.976	a about	2.247.410	2 443.740		3.404.088
14.0	= 0	orts.		20	307.418	304.350	304.284	442.816	428,508	428.013	507,074	501,567	501,260
				lbs,	13,570	14,532	15,567	16,102	17,953	18,490	18,004		21,724
16 1.0		Head diagonals, (589), (590)		<u>6</u>	\$0,204	19,849	19,608	29,443	28,856	28,436	41.74		40,166
17 1.0	•	2		lbs	5,991,350	5,088,489		8,674,540	8,667,056	8,675,865	12,255-920	-	\$\$046be#1
9	0	Bridge weight per running-foot .		ă	7+489	7,485	7,496	9,638	9.630	9,639	12,256		648,51
19 1.0		Panel Rugth	*+*	feet	454	4319	9	ያነ	4719	45	555	Set de	2,
	: : Q	Ratio of length to central height,	* + *		4.915	4.897	4.240	3.996	900+	4.015	3.780		3-797
0': 1#		Katso of munimum ocad to live	77. 2			·	•	. 6	0	9	7		,
*	=	Ratio of minimum dead to total	7		3.7440	3-74-96	3-7430	4.6193	4-6150	4.6199	048340	D.1905	0.1845
		peol			0.7892	0.7891	97.0	0,8282	o.Baßo	0.8082	0.8997	a.8596	0.8597
91	3	107- rok J Best number of nanels.	+ 0										
2	=	Bridge weight	**	tons								•	
_		nels]			16	17	18	16	17	18	91		18
		ت	*	3	303.000	30,001	m	340.059	239.174	#38.295	#79.688	278.814	277.95E
20,1,5	: :	Least bridge weight	M IS	<b>2003</b>	3,580.500		3,582.758	5,114,320		5,114,813	7,130.005		7,132,061
80 200		~	R -4	į	aye yes	100	212 824	B99 036	OAD ARR				allo foo
90	;	Least bridge weight   tancous,	* 15*	tons	4,145-736		4,140.448	5,867.303	5,861 533	5,873.936	810	8,095-976	8,000,508
_	í	Township of a man a land	<b>u</b> ~	į.				•					
, ;	) = =	f amiting spain, I	. ^										
27	L = 0	Luniting span, to see	. ~	13									
_					_	_	_						

183. Among the deductions to be drawn from this table, for the Brunel double-bow bridge of double web system, are the following:—

1st, For a given uniform live load,

$$n \propto l^{\frac{1}{2}}$$
 nearly; (620)

and generally

$$n \propto \left(\frac{l}{nL}\right)^{\frac{1}{3}} \times l^{\frac{1}{3}}$$
 nearly. (621)

2d, For spans less than 400 feet,

$$h = \frac{l}{4.8} \left(\frac{nL}{l}\right)^{\frac{1}{2}} \text{ nearly.}$$
 (622)

For spans of 400 feet and upwards,

$$h = \frac{l}{4 \cdot 3} \left(\frac{nL}{l}\right)^{\frac{1}{3}} \text{ nearly.}$$
 (623)

3d, For different spans with same live load per running-foot,

$$W \propto lh \text{ nearly};$$
 (624)

and for the same span under different uniform live loads,

$$nW \propto \left(\frac{nL}{l}\right)^{\frac{1}{2}}$$
 nearly. (625)

Many other conclusions may be drawn from this table, and weights of intermediate spans may be derived by interpolation; but the equations (598) to (619), inclusive, cover the whole case.

184. Example. — We now proceed to find the strain sheet for the 200-feet span of the table in article 182 in a manner similar to that employed in article 169.

We now have

$$l = 200$$
 feet,  $q = 16$  feet;  
 $h = 40.788$  feet,  $q_1 = 18$  feet.  
 $n = 13$ .  
 $nL = 200$  tons,  $L = 15\frac{5}{13}$  tons;  
 $nW = 103.288$  tons,  $W = 7.945$  tons;  
 $W + L = 23.330$  tons.

Weight of floor, by article 174,  $\frac{530}{8} \times 200 = 34667$  pounds; Total live load, nL, = 400000 pounds. Total load on longitudinal **I**-beams = 434667 pounds.

Load on each panel length of every longitudinal I-beam spaced 3.2 feet

$$=\frac{3.2}{16}\times\frac{434667}{13}=6687$$
 pounds.

Then, by (502),

Cross-section of beam

$$= S = 0.00015 \times \frac{434667}{13} = 5.0154$$
 square inches.

Take b = 4 inches = breadth of flange.  $b - b_1 = 0.26$  inch = thickness of web. Then, from (552), (551), and (550),

$$d = 9.774$$
 inches = depth of beam,  
 $d - d_1 = 0.661$  inch = depth of two flanges,  
 $I = 75.416$  = moment of inertia of section,

which is larger than I for the sections given by ordinary beams of the same area of section.

Weight of longitudinal I-beams, 6 in number, =  $6 \times \frac{5}{18} \times 12 \times 200 \times 5.01538 = 20062$  pounds.

## Upon the transverse I-beams we have

Live load, 400000 pounds, Floor, 34667 pounds, Longitudinal I-beams, 20062 pounds.

Total for 13 panels, 454729 pounds.

Load on 1 beam, 34979 pounds.

From (504),

$$\frac{I}{d} = \frac{12 \times 18 \times 34979}{8 \times 2 \times 10000} = 47.2216 = 2S,$$

by (505);

S = 23.6108 square inches for vertical load, and for the wind pressure,

$$W_1 = 2500 \times \frac{40.788}{13} = 7844 \text{ pounds per panel,}$$

$$Q_2 = \frac{8000}{1 + 0.93312 \times \frac{13}{200}} = 7542.5 \text{ pounds per square inch,}$$

$$S = \frac{2 \times 7844}{3 \times 13 \times 7542.5} (12^2, 11^2, 10^2, 9^2, 8^2, 7^2) \text{ by (511);}$$

$$= 7.6798 \text{ square inches, 1st and 12th beams;}$$

$$= 6.4532 \text{ square inches, 2d and 11th beams;}$$

$$= 5.3332 \text{ square inches, 3d and 10th beams;}$$

$$= 4.3199 \text{ square inches, 4th and 9th beams;}$$

$$= 3.4132 \text{ square inches, 5th and 8th beams;}$$

$$= 2.6133 \text{ square inches, 6th and 7th beams.}$$

The total cross-sections for each half-span are

S = 31.2906 square inches, = 30.0640 square inches, = 28.9440 square inches, = 27.9307 square inches, = 27.0240 square inches, = 26.2241 square inches. Satisfying the condition (505), we may assign values to d and  $d_1$ , and use (557) and (558) in finding the thickness of each transverse beam.

Put 2 light 12-inch beams at each panel point, the section of each being  $\frac{1}{2}S$ . Then we have

$$d = 12$$
 inches = depth of beam,  
 $d - d_1 = 2$  inches = depth of 2 flanges;

and (558) becomes, for breadth of flanges,

$$b = \left(\frac{24 - 12}{2 \times 22} + \frac{1}{12}\right) \times \frac{1}{2}S = 0.17803S = 5.5707 \text{ inches,}$$

$$= 5.3523 \text{ inches,}$$

$$= 5.1529 \text{ inches,}$$

$$= 4.9725 \text{ inches,}$$

$$= 4.8111 \text{ inches,}$$

$$= 4.6687 \text{ inches;}$$

and (557) gives

$$b_1 = \frac{(24 - 12)12}{10 \times 2 \times 22} \times \frac{1}{2}S = 0.16363S = 5.1203 \text{ inches,}$$

$$= 4.9083 \text{ inches,}$$

$$= 4.7363 \text{ inches,}$$

$$= 4.4221 \text{ inches,}$$

$$= 4.2912 \text{ inches.}$$
Thickness of web = 0.4504 inch =  $b - b_1$ ,
$$= 0.4440 \text{ inch,}$$

$$= 0.4166 \text{ inch,}$$

$$= 0.4020 \text{ inch,}$$

$$= 0.3890 \text{ inch,}$$

$$= 0.3775 \text{ inch.}$$

The weight of these 24 transverse I-beams is

$$12 \times 18 \times \frac{5}{18}\Sigma S = 20577$$
 pounds,

since  $\Sigma S$  (= sum of all the cross-sections) is 342.9548 square inches.

Cross-sections of horizontal diagonals are found by dividing the strains in (508) by 15,000, where we now have

$$IV_1 = 7844, \quad \frac{IV_1}{15000} = 0.52293,$$
  
 $\sin \phi_1 = 0.76017,$ 

$$S = \frac{0.52293}{2 \times 13 \times 0.76017} (13 \times 12, 12 \times 11,$$

$$11 \times 10, 10 \times 9, 9 \times 8, 8 \times 7, 7 \times 6)$$

$$= 0.026458 \times 156 = 4.1274 \text{ square inches,}$$

$$= 0.026458 \times 132 = 3.4925 \text{ square inches,}$$

$$= 0.026458 \times 110 = 2.9104 \text{ square inches,}$$

$$= 0.026458 \times 90 = 2.3812 \text{ square inches,}$$

$$= 0.026458 \times 72 = 1.9050 \text{ square inches,}$$

$$= 0.026458 \times 56 = 1.4816 \text{ square inches,}$$

$$= 0.026458 \times 42 = 1.1112 \text{ square inches,}$$

for the respective panels.

$$\therefore$$
  $\Sigma S = 67.4150$  square inches,

and weight of 26 horizontal diagonals is

$$\frac{12 \times 18}{\sin \phi_1} \times \frac{5}{18} \times 67.415 = 5321$$
 pounds.

The cross-section of each panel length of each wind chord is given by (514), thus,

$$S = \frac{7844 \times 200}{2 \times 13 \times 18 \times 6400} (1 \times 12, 2 \times 11, 3 \times 10, 4 \times 9, 5 \times 8, 6 \times 7, 7 \times 6)$$

$$= 0.52377 \times 12 = 6.2852 \text{ square inches,}$$

$$= 0.52377 \times 22 = 11.5229 \text{ square inches,}$$

$$= 0.52377 \times 30 = 15.7131 \text{ square inches,}$$

$$= 0.52377 \times 36 = 18.8557 \text{ square inches,}$$

$$= 0.52377 \times 40 = 20.9508 \text{ square inches,}$$

$$= 0.52377 \times 42 = 21.9983 \text{ square inches,}$$

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$$= 0.52377 \times 42 = 21.9983 \text{ square inches.}$$

These sections can easily be made up of channels and plates, or of beams and plates, with the required radius of gyration given in article 162.

In summing these sections for the weight formula, all are to be taken four times, except the last, which is taken twice only.

Weight of wind chords = 
$$\frac{12 \times 200 \times 106.3253}{13} \times \frac{5}{18} = 21810 \text{ pounds.}$$

Supported by verticals, we have

Live load, 400000 pounds, Floor, 34667 pounds, Longitudinal I-beams, 20062 pounds, Horizontal diagonals, 5321 pounds, Wind chords, 21810 pounds.

Weight on each vertical = 
$$\epsilon_{\pi}$$
 = 19390

Therefore we have the cross-sections, by (580),

$$S = \frac{19390}{10667} = 1.81776$$
 square inches;  
by (582),  
 $S_1 = \frac{19390}{12000} = 1.61583$  square inches;

for the lower and upper halves respectively of the verticals due load; and, for the bending-moment due wind, (584) gives

$$S_2 = \frac{15 \times 200}{68 \times 13} = 3.39367$$
 square inches,

Section of compressed half = 5.21143 square inches, Section of extended half = 5.00950 square inches.

And, since the upper and lower halves of the girder are symmetrical, and the sum of the lengths of the verticals  $= \sum y = \frac{1}{3}h\left(n - \frac{1}{n}\right) \text{ by (521)}, = \frac{1}{3} \times 40.788 \times \frac{16.8}{13}, \text{ we have}$ 

Weight of lower halves

$$= 2 \times \frac{5}{18} \times 5.21143 \times \frac{12}{3} \times 40.788 \times \frac{168}{13} = 6103$$
 pounds,

Weight of upper halves

$$= 2 \times \frac{5}{18} \times 5.0095 \times \frac{12}{3} \times 40.788 \times \frac{168}{13} = 5866$$
 pounds.

Total weight

$$= 11969 \times \frac{11}{10}$$

= 13165 pounds.

after adding  $\frac{1}{10}$  for braces, etc.

The sections may be made up of 2 channels, the one vertical, the other inclined at an angle whose tangent is  $\frac{1}{10}$ .

According to the principles of article 165, the bars in the bracing of these supports should have a cross-section of about  $\frac{1}{2}$  inch; that is, about  $\frac{1}{10}$  of  $(S + S_2)$ .

Equation (587) gives the cross-section of each head diagonal thus:

$$S = \frac{0.0625 \times 168 \times 40.788 \times 200}{6 \times 13^3 \times 18 \times 0.64972} = 0.5556$$
 square inch.

From (589) comes the weight of 14 head diagonals in the seven central panels equal to

$$14 \times \frac{5}{18} \times \frac{12}{13} \times \frac{200}{0.64972} \times 0.5556 = 614$$
 pounds.

Cross-section of head struts, by (591),

$$=\frac{168 \times 40.788}{12 \times 169} = 3.3789$$
 square inches,

requiring 2 light 4-inch channels latticed not less than 4 inches apart.

Weight of 8 head struts,

$$\frac{11}{10} \times 8 \times \frac{5}{18} \times 12 \times 18 \times 3.3789 = 1784$$
 pounds,

after adding  $\frac{1}{10}$  for bracing.

Increment of section of top chord due to head diagonal strain is given by (595), thus:

$$S = \frac{156.25 \times 168 \times 40.788 \times 200}{7529 \times 13^3 \times 18} = 0.7192 \text{ square inch,}$$

the strain being = 0.7192  $\times \frac{7529}{2000} = 2.707$  tons.

Weight added to top chords

= 
$$2 \times 7 \times \frac{5}{18} \times \frac{12 \times 200}{13} \times 0.7192 = 516$$
 pounds.

For each of two girders, the horizontal component of chord strain is, by (564),

$$H = \frac{1}{2} \times 23.330 \times \frac{13 \times 200}{8 \times 40.788} = 92.947 \text{ tons};$$

and the chord strains are

$$P = \frac{9^2.947}{\cos \alpha} = U = \frac{9^2.947}{\cos \beta} = 99.317 \text{ tons, 1st panel;}$$

$$= 97.415 \text{ tons, 2d panel;}$$

$$= 95.830 \text{ tons, 3d panel;}$$

$$= 94.580 \text{ tons, 4th panel;}$$

$$= 93.672 \text{ tons, 5th panel;}$$

$$= 93.130 \text{ tons, 6th panel;}$$

$$= 92.947 \text{ tons, 7th panel.}$$

Cross-section of top chord due load

Augment 4th, 5th, 6th, 7th, 8th, 9th, 10th, by 0.7192.

```
Cross-section of bottom chord = \frac{U}{5} = 19.863 square inches, 1st panel;

= 19.483 square inches, 2d panel;

= 19.166 square inches, 3d panel;

= 18.916 square inches, 4th panel;

= 18.734 square inches, 5th panel;

= 18.626 square inches, 6th panel;

= 18.589 square inches, 7th panel.
```

The top chord may be composed of 2 9-inch channels and 1 plate; the bottom chord, of 3 bars and 2 bars in alternate panels.

From (561), we find

$$\Sigma \sec^2 \alpha = 27.4329;$$

and (566) gives, adding  $\frac{1}{10}$ ,

Weight of top chords = 
$$\frac{11}{10} \times \frac{3}{2} \times \frac{5}{18} \times \frac{11.665}{3.7647} \times \frac{200^2}{40.788} \times 27.4329$$
  
= 38206 pounds.

From (567),

Weight of bottom chords =  $\frac{3.7647}{5}$  × 38206 = 28767 pounds.

From (571), the strain on a girder diagonal is called

```
Z = \frac{\frac{1}{4} \times 15 \frac{5}{13} \times 200 \times 1.8}{2 \times 8 \times 40.788} \times \sec \theta = 2.12166 \sec \theta,
   = 3.1023 tons, 2d panel. Section = \frac{8}{3}Z = 1.16 square inches.
   = 4.0600 tons, 3d panel.
                                                     = 1.52 square inches.
                                                     = 1.83 square inches.
   = 4.8790 tons, 4th panel.
   = 5.4861 tons, 5th panel.
                                                     = 2.06 square inches.
   = 5.8564 \text{ tons},
                      6th panel.
                                                     = 2.20 square inches.
                                                     = 2.24 square inches.
   = 5.9807  tons,
                      7th panel.
   = 5.8564 tons, 8th panel.
                                                     = 2.20 square inches.
   = 5.4861 tons, 9th panel.
                                                     = 2.06 square inches.
  = 4.8790 tons, 10th panel.
                                                     = 1.83 square inches.
  = 4.0603 tons, 11th panel.
                                                     = 1.52 square inches.
                                                     = 1.16 square inches.
  = 3.1023 tons, 12th panel.
```

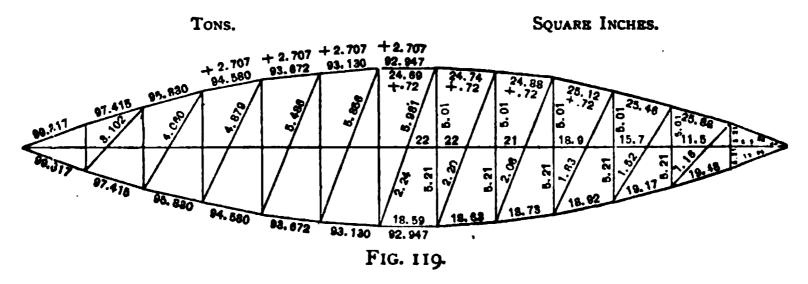
These sections, being in alternate tension and compression, may be made up of 4 angle irons,  $1\frac{1}{2} \times 1\frac{1}{2}$ , latticed at such a distance apart that the unsupported length may be not more than one hundred times the radius of gyration of the section.

The weight of these girder diagonals is, from (572), equal to

$$\frac{4 \times 12 \times 5 \times 0.45 \times 200^{3} \times 3}{8 \times 13^{2} \times 18 \times 40.788 \times 8} \times \sum \sec^{2} \theta \times \frac{11}{10} = 326.41 \sum \sec^{2} \theta \times \frac{11}{10} = 21088 \text{ pounds,}$$

since  $\Sigma \sec^2 \theta = 58.7324$  by (563), and we increase by one-tenth for latticing and attachments.

### STRAINS AND CROSS-SECTIONS.



For each of two girders. Span, 200 feet. Central height, 40.788 feet. Uniform live load, 1 ton = 2,000 pounds per linear foot, applied at centres of verticals; the verticals in this case acting merely as struts in the lower half, and as suspenders in the upper half. The diagonals, only one-half of them being shown in the figure, are alternately in tension and compression; the greatest strains being the same on each of the two diagonals of a panel. Bridge weight = 103.288 tons.

The deflection due to full load is found for any point by equation (559), having now

$$B_1 = 4.380$$
 tons per square inch,  
 $E = 12000$  tons per square inch,  
 $h_1 = h = 40.788$  feet,  
 $a = \frac{1}{2}l = 100$  feet,

and x being measured from centre of span;

```
.. Deflection D_1 = 1.490 inches for x = 0, centre;

D_2 = 1.483 inches for x = 100 \div 13;

D_3 = 1.432 inches for x = 300 \div 13;

D_4 = 1.327 inches for x = 500 \div 13;

D_5 = 1.162 inches for x = 700 \div 13;

D_6 = 0.815 inch for x = 900 \div 13;

D_7 = 0.583 inch for x = 1100 \div 13;

D_8 = 0 inch for x = 1300 \div 13, ends.
```

Equation (366) yields the excess of length required in top chord to give the proper camber,

$$\lambda = \frac{3.76 + 5}{12000} \times 205.34 \times 12 = 1.8$$
 inches,

since length of polygonal top chord is equal to

$$\frac{200}{13} \times \sum \sec \alpha = \frac{200}{13} \times 13.3466 = 205.34 \text{ feet};$$

... Mean excess per panel =  $\frac{1.8}{13}$  = 0.138 inch,

or a little more than  $\frac{1}{8}$  inch.

# CHAPTER XI.

BRIDGES OF CLASS II. — BEST NUMBER OF PANELS AND BEST HEIGHT DETERMINED FOR A GIVEN SPAN UNDER A GIVEN UNIFORM LIVE LOAD. — LEAST BRIDGE WEIGHT AND LIMITING SPAN FOUND.

### SECTION 1.

The Parabolic Bowstring Girder of Double Triangular System (Fig. 35), with the Extreme Diagonals omitted, and a Vertical Suspender at Extreme Panel Point.

185. Let l = span, in feet.

h =height of girder at centre, in feet.

n = number of panels.

L =panel weight of uniform live load, in tons.

W = panel weight of bridge, in tons.

The height of girder at any point, x, is given by (472), and at all vertices by (473), if we make r = 1 for the first point, and put 2h for h throughout, thus:

$$y_r = \frac{4h}{n^2}r(n-r), \qquad (626)$$

$$y_{r+1} = \frac{4h}{n^2}(r+1)(n-r-1),$$

$$\Delta y = y_{r+1} - y_r = \frac{4h}{n^2}(n-2r-1),$$
 (627)

$$\tan \alpha = \Delta y + \frac{l}{n} = \frac{4h}{nl}(n - 2r - 1),$$
 (628)

$$\sec^2 \alpha = 1 + \tan^2 \alpha = 1 + \frac{16h^2}{n^2\ell^2}(n - 2r - 1)^2,$$
 (629)

$$\tan \theta_r = -\tan \phi_{r-1} = -y_r \div \frac{l}{n} = -\frac{4h}{nl}r(n-r),$$
 (630)

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{16h^2}{n^2/2}r^2(n-r)^2. \tag{631}$$

 $\alpha$ ,  $\theta$ , and  $\phi$  are defined in article 49.

The live load and a large part of the dead load are applied at the panel points of the bottom chord, and are transmitted by the diagonals to the parabolic arch, which is equilibrated by the uniform load, leaving only a tensile strain on the diagonals from full uniform load. We shall assume that the two diagonals which support any panel weight of uniform dead load carry each one-half of the same.

186. Moments at all panel points due n(W + L), the total load, are, from equation (65),

$$M_r = \frac{W + L}{2n}l(n-r)r;$$

and the horizontal component of chord strain under same load is equal to

$$H=\frac{M}{y}=\frac{1}{8}(W+L)\frac{nl}{h},$$

as in (564), and is uniform throughout for maximum.

Greatest strains in top chord 
$$=P=\frac{H}{\cos\alpha}$$
, (632)  
Greatest strains in bottom chord  $=U=H$ 

Cross-section of top chord, S = P + QCross-section of bottom chord,  $S_1 = U + T$  }, (633)

Take 
$$Q = \frac{4}{1 + \frac{50^2}{40000}} = 3.7647 \text{ tons}$$
 (634)

as the allowed inch strains on top and bottom chords respectively. Length of segment of top chord =  $\frac{12l}{n\cos\alpha}$  inches,

Volume of segment of top chord

$$= \frac{12lH}{nQ\cos^2\alpha} = \frac{3}{2}(W+L)\frac{l^2}{Qh\cos^2\alpha}$$
 cubic inches,

$$\sum \sec^2 \alpha = n + \frac{16h^2}{3l^2} \left(n - \frac{1}{n}\right),$$
 (635)

by summing (629) for values of r from 0 to n-1, inclusive.

Therefore, calling weight of a cubic inch of wrought-iron, as in all cases,  $\frac{5}{18}$  pound, we find

Weight of top chords, in pounds,

$$= \frac{3}{2} \times \frac{5}{18} (W + L) \frac{l^2}{Qh} \sum \sec^2 \alpha, \qquad (636)$$

$$= \frac{5(W + L)l^2}{12 \times 3.7647h} \left\{ n + \frac{16h^2}{3l^2} \left( n - \frac{1}{n} \right) \right\}$$

$$= \frac{W + L}{h} \left\{ 0.1106771nl^2 + 0.59028 \left( n - \frac{1}{n} \right) h^2 \right\}$$

$$= \frac{W + L}{h} \left\{ 0.885417l^2 + 4.64844h^2 \quad 8$$

$$0.996095l^2 + 5.24688h^2 \quad 9$$

$$1.106772l^2 + 5.84372h^2 \quad 10$$

$$1.217449l^2 + 6.43936h^2 \quad 11$$

$$1.328126l^2 + 7.03412h^2 \quad 12$$

$$1.438804l^2 + 7.62820h^2 \quad 13$$

$$1.549481l^2 + 8.22168h^2 \quad 14$$

$$1.660158l^2 + 8.81480h^2 \quad 15$$

$$1.770835l^2 + 9.40752h^2 \quad 16$$

$$1.881512l^2 + 10.00000h^2 \quad 17$$

$$1.992190l^2 + 10.59216h^2 \quad 18$$

$$2.102867l^2 + 11.18440h^2 \quad 19$$

$$2.213544l^2 + 11.77600h^2 \quad 20$$

Weight of bottom chords, in pounds, =  $\frac{5}{18} \times \frac{12}{8} \frac{lV + L}{Th} \times nl^2$ , (637) =  $\frac{W + L}{h} \times \frac{l^2n}{12}$ 

$$= \frac{IV + L}{h}$$

$$0.666667/^{2}$$

$$0.750000/^{2}$$

$$0.833333/^{2}$$

$$0.916667/^{2}$$

$$1.000000/^{2}$$

$$1.083333/^{2}$$

$$1.166667/^{2}$$

$$1.250000/^{2}$$

$$1.333333/^{2}$$

$$1.416667/^{2}$$

$$1.500000/^{2}$$

$$1.583333/^{2}$$

$$1.583333/^{2}$$

$$1.1666667/^{2}$$

$$1.250000/^{2}$$

$$1.250000/^{2}$$

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187. The Girder Diagonals. — Separating the double system into the single web systems of Fig. 35a and Fig. 27, let us consider first that of Fig. 35a, and find the difference,  $\Delta H$ , of horizontal strains at the foremost end of the advancing uniform discontinuous load, nL, and for the same instant at the next two forward panel points of the double system.

Putting L for W in equation (60), and taking  $r_2 = -\frac{1}{2}$ , we find the moment at any point, x, at or before the foremost end, to be

$$M_x = \frac{Lc}{2l}(r + \frac{1}{2})^2(l - x),$$
 (638)

where c = length of whole interval in the single system  $= \frac{2l}{n}$  in the double system,  $r + \frac{1}{2} = \text{number of panel points of bottom chord loaded}$ , x = distance from left end to the point where moment is taken.

From (472), putting 2h for h,

$$y = \frac{4h}{l^2}x(l-x). {(639)}$$

Therefore the simultaneous horizontal strains due live load at the three consecutive panel points, of which the foremost end of live load is at the rear one, are

$$H = \frac{M_x}{y} = \frac{Llc}{8h} \cdot \frac{(r + \frac{1}{2})^2}{x},$$

$$= \frac{Ll}{8h} \frac{(r + \frac{1}{2})^2}{r} \text{ if } x = rc, \text{ foremost end;}$$

$$= \frac{Ll}{8h} (r + \frac{1}{2}) \text{ if } x = (r + \frac{1}{2})c, \text{ next panel point;}$$

$$= \frac{Ll}{8h} \frac{(r + \frac{1}{2})^2}{r + 1} \text{ if } x = (r + 1)c, \text{ next panel point;}$$

where r takes the successive values  $\frac{1}{2}$ ,  $\frac{3}{2}$ ,  $\frac{5}{2}$ ,  $\frac{7}{2}$ , ...  $\frac{n-1}{2}$ .

Then we find the greatest differences, that is, the greatest horizontal component of diagonal strain due live load, thus:

$$\Delta_{1}H = \frac{Ll}{8h} \left\{ r + \frac{1}{2} - \frac{(r + \frac{1}{2})^{2}}{r} \right\} = -\frac{Ll}{16h} \frac{r + \frac{1}{2}}{r}$$

$$\Delta_{1}H = \frac{Ll}{8h} \left\{ \frac{(r + \frac{1}{2})^{2}}{r + 1} - (r + \frac{1}{2}) \right\} = -\frac{Ll}{16h} \frac{r + \frac{1}{2}}{r + 1}$$
(641)

for the first single system; r taking the successive values  $\frac{1}{2}$ ,  $\frac{3}{2}$ ,  $\frac{4}{2}$ , etc.

Similarly, for the second single system, we get greatest horizontal component of diagonal strain due live load,

$$\Delta_{1}H = -\frac{Ll}{16h}\frac{r+1}{r+\frac{1}{2}}$$

$$\Delta_{1}H = -\frac{Ll}{16h}\frac{r}{r+\frac{1}{2}}$$

$$, (642)$$

where r becomes 1, 2, 3, 4, etc.

Now, if we consider the horizontal components of the two diagonals in the same panel of the double system, Fig. 35, with its extreme tie made vertical, we see that  $\Delta_1 H$  of (641) and  $\Delta_2 H$  of (642) belong to the odd panels, and  $\Delta_2 H$  of (641) and  $\Delta_1 H$  of (642) belong to the even panels. Also, for the odd panels, r of (641) is less by  $\frac{1}{2}$  than r of (642); and, for the even panels, r of (641) is greater by  $\frac{1}{2}$  than r of (642).

Therefore, reducing so that r belongs to the second system, we find

Compression, 
$$\Delta_1 H = -\frac{Ll}{16h} \frac{r}{r + \frac{1}{2}}$$
 odd panels, (643)  
Tension,  $\Delta_1 H = -\frac{Ll}{16h} \frac{r + 1}{r + \frac{1}{2}}$ 

Tension, 
$$\Delta_{\frac{1}{2}}H = -\frac{Ll}{16h}\frac{r+1}{r+\frac{1}{2}}$$
 even panels, (644)  
Compression,  $\Delta_{1}H = -\frac{Ll}{16h}\frac{r}{r+\frac{1}{2}}$ 

which expressions are identical; and the sum of either pair is

$$\Delta_1 H + \Delta_{\frac{1}{2}} H = -\frac{Ll}{16h} \frac{2(r+\frac{1}{2})}{r+\frac{1}{2}} = -\frac{Ll}{8h},$$
 (645)

as already given by (570) for the total horizontal component of diagonal strain due live load in any panel.

Since these diagonals are to be alternately in tension and compression, the load travelling either way, and our specifications would multiply the compressive strains by 1.8, we shall, for convenience, take

$$\Delta H = -\frac{Ll}{8h}$$

due live load for each diagonal in a panel, instead of multiply-

ing by 1.8, and shall treat all diagonals as in compression under the inch strain,

$$Q = \frac{4}{1 + \frac{100^2}{20000}} = \frac{8}{3}$$
 tons.

This procedure varies a little from the specifications, but on the safe side, since

$$2r + 1 > 1.8r$$
.

Moreover, since the dead load, with the exception of the top chords and head system and wind braces, is suspended at all times on two diagonals which transmit it to the equilibrated top chord, and since the tensile section will, for practical spans, not be greater than the compressive section due to live load as above augmented, and to be provided for in compression, we may, as appears below, leave the tensile strain which will come upon the diagonals acting as suspenders, almost entirely to the material put into them to resist maximum compression.

It is to be observed, that, when the live load is fully on the bridge, there is no compression on the girder diagonals, but each one acts simply as a suspender to transmit  $\frac{1}{2}(W_2 + L)$  to the equilibrated top chord;  $W_2$  being that part of the dead load at any lower apex. Now, from the results tabulated in Chap. X., we may doubtless, in the present case, for spans not over 600 feet, consider  $W_2$  as ranging from  $\frac{4}{5}W$  to  $\frac{2}{3}W$ , while W ranges from  $\frac{1}{3}L$  to  $\frac{2}{5}L$  nearly, in spans from 100 feet to 600 feet.

Taking the greater,  $\frac{7}{6}L$ , as the vertical component of tension on each girder diagonal, we have

$$\frac{7}{6}L\cot\theta = \frac{7}{24} \times \frac{Ll}{h} \times \frac{n}{r(n-r)}$$

as the horizontal component of tension on diagonals acting as

suspenders; the greatest value of which is found when r = 1, or r = n - 1. That is,

$$(\Delta H)_{\text{max}} = \frac{7}{24} \cdot \frac{Ll}{h} \cdot \frac{n}{n-1}$$
 for tension.

But

$$\Delta H = -\frac{Ll}{8h}$$
 for compression.

Dividing  $\Delta H_{\text{max}}$  and  $\Delta H$  by 5 and by  $\frac{8}{8}$ , the allowed inch strains in tons respectively for tension and compression, and multiplying by  $\sec \theta$ , we find these resulting cross-sections for comparison:

$$S_{i} = \frac{1}{17} \times \frac{Ll}{h\cos\theta} \times \frac{n}{n-1} \text{ nearly}$$

$$S = \frac{3}{64} \times \frac{Ll}{h\cos\theta}$$
(646)

Now,  $S_1$  will be greater than S only when r = 1, or r = n - 1; that is, the girder diagonals which meet at the first and second and at the  $(n-2)^{th}$  and  $(n-1)^{th}$  lower apices, will need additional section under full load to the extent of the difference between  $S_1$  and S. And this we shall supply in the vertical braces at these points.

$$\therefore \text{ Cross-section of a girder diagonal} = S = \frac{3Ll}{64h\cos\theta}; \quad (647)$$

Weight of 2(n-2) girder diagonals, in pounds,

$$= 2 \times \frac{12l^2}{n} \times \frac{5}{18} \times \frac{3}{64} \times \frac{L}{h} \times \sec^2 \theta, \qquad (648)$$

$$= \frac{5Ll^2}{16nh} \times \sec^2 \theta$$

$$= \frac{5Ll^2}{16nh} \left\{ n - 2 + \frac{16h^2}{n^2l^2} \left( \frac{n^5}{30} - n^2 + \frac{59}{30}n - 1 \right) \right\}$$

$$= \frac{L}{h} \left\{ \frac{5(n-2)l^2}{16n} + \left( \frac{n^2}{6} - \frac{5}{n} + \frac{59}{6n^2} - \frac{5}{n^3} \right) h^2 \right\}$$

$$= \frac{L}{h} \quad 0.234375^{2} + 10.18555h^{2} \quad 8$$

$$0.243056^{2} + 13.05899h^{2} \quad 9$$

$$0.250000^{2} + 16.26000h^{2} \quad 10$$

$$0.255682^{2} + 19.78964h^{2} \quad 11$$

$$0.260417^{2} + 23.64873h^{2} \quad 12$$

$$0.264423^{2} + 27.83796h^{2} \quad 13$$

$$0.267857^{2} + 32.35788h^{2} \quad 14$$

$$0.270833^{2} + 37.20889h^{2} \quad 15$$

$$0.273437^{2} + 42.39136h^{2} \quad 16$$

$$0.275735^{2} + 47.90556h^{2} \quad 17$$

$$0.277778^{2} + 53.75172h^{2} \quad 18$$

$$0.279605^{2} + 59.93002h^{2} \quad 19$$

$$0.281250^{2} + 66.44062h^{2} \quad 20$$

Adding together (636), (637), and (648), we find

Weight of girders due to loads, pounds,

•			•	12
$=\frac{L}{h}$	$1.786459l^2 + 14.83399h^2$	$+\frac{lV}{h}$	$1.552084l^2 + 4.64844h^2$	8
	$1.989151/^2 + 18.30587h^2$		$1.746095l^2 + 5.24688h^2$	9
	$2.190105l^2 + 22.10372h^2$		$1.940105l^2 + 5.84372h^2$	10
	$2.389798l^2 + 26.22900h^2$	1	$2.134116l^2 + 6.43936h^2$	11
	$2.588543l^2 + 30.68285h^2$		$2.328126l^2 + 7.03412h^2$	12
	$2.786560l^2 + 35.46616h^2$		$2.522137l^2 + 7.62820h^2$	13
	$2.984005l^2 + 40.57956h^2$		$2.716148/^2 + 8.22168h^2$	14
	$3.180991l^2 + 46.02369h^2$		$2.910158l^2 + 8.81480h^2$	15
	$3.377605l^2 + 51.79888h^2$		$3.104168l^2 + 9.40752h^2$	16
	$3.573914l^2 + 57.90556h^2$		$3.298179l^2 + 10.00000h^2$	17
	$3.769968l^2 + 64.34388h^2$		$3.492190l^2 + 10.59216h^2$	18
	$3.965805l^2 + 71.11442h^2$		$3.686200l^2 + 11.18440h^2$	19
	$4.161461l^2 + 78.21662h^2$		$3.880211l^2 + 11.77600h^2$	20

To be augmented by one-tenth.

188. Floor to be the same as in article 174.

Weight of floor = 
$$\frac{520}{3}$$
 pounds. (649)

- 189. Take longitudinal I floor beams, as in articles 157, 175, equation (575).
- 190. The transverse I-beams supporting live load, floor, and longitudinal beams are here conditioned as in article 158, and their cross-section due vertical forces is given by equation (506). Their weight due same forces is given by (576).

The cross-sections of transverse I-beams due to wind are expressed in (511).

The weight of iron to be added to the transverse I-beams on account of wind, is given by (512) and (577) for n even and n odd.

The whole effect of wind pressure is to be transferred to the horizontal system, in the plane of the bottom chords, by means of vertical braces connecting each transverse beam with both top chords.

It may be observed, that in this and all like cases a shearing-stress is generated throughout the transverse beam by the wind pressure transmitted through these vertical braces; but there will be sufficient reserve material in the web of the beam to resist this shearing-stress, as becomes evident on reflection.

- 191. If we divide equation (508) by 15,000, we have the cross-section of any horizontal diagonal in the floor system. And equations (509) and (578) give us the weight of the horizontal diagonals, in pounds, for the even and odd values of n.
- 192. For wind chords, let us use the bottom chords of the girders, augmenting their cross-sections by the quantity in (513) divided by the tensile inch strain, 10,000 pounds.

Although this augment will only resist tension, while the compressive chord strain due to wind will sometimes be greater than the tensile chord strain due to dead load, yet, as the

excess of compression is not great, it may be left safely to the outside longitudinal I-beams, which, it will be remembered, are otherwise only half loaded.

We may compare the chord strains due dead load and due wind by means of Fig. 112, thus:

For 
$$W_i$$
,  $N = \frac{W_l}{2nh}$ ;  
For  $W_i$ ,  $N = \frac{W_i l}{2nq_i}$ .

These co-efficients of strain will be equal when

$$q_1 = \frac{W_1 h}{W}; (650)$$

and this might be made a condition determining the width,  $q_1$ , of the bridge, so that no compressive strain would prevail in a bottom chord. But we shall not now change the uniform value of  $q_1 = 18$ , assumed at first.

Therefore the increase of section of each bottom chord due to wind, as derived from (513), is, in square inches,

$$S = \frac{W_1 l}{20000nq_1} \{ (n-1), 2(n-2), 3(n-3), \text{etc.} \}. \quad (651)$$

The weight of this wind augment to bottom chords is found by putting 10,000, instead of 6,400, for Q in equations (515) and (579), thus:

Weight of bottom chords due to wind, pounds,

$$= 0.00385802496 \left( 2 + \frac{3}{n} - \frac{2}{n^2} \right) h l^2 \qquad (n \text{ even})$$

$$= 0.00385802496 \left( 2 + \frac{3}{n} - \frac{2}{n^2} - \frac{3}{n^3} \right) h l^2 \qquad (n \text{ odd})$$

$= hl^2$	0.0090422	n = 8	$\  = hl^2 \ $	0.0084499	n =	15
	0.0088909	9		0.0084093		16
	0.0087963	10		0.0083678		17
	0.0086957	11		0.0083352		18
	0.0086272	12		0.0083021		19
1	0.0085555	13		0.0082755		20
	0.0085034	14				

193. Assuming that a wind pressure per panel of  $125\frac{l}{n}$  pounds (l being in feet) acts in a direction normal to the plane of girder at each apex in each top chord, we have the moment of each brace due wind equal to

$$125\frac{l}{n} \times y = \frac{1}{2}S \times \frac{1}{10}B_1y$$
 (653)

if S = cross-section of flanges of a brace,  $B_r =$  inch strain allowed for bending-moment  $= \frac{1}{2}(5,333 + 6000) = \frac{17000}{3}$ , and if the length of brace is to its width at broadest end in the ratio of 10 to r, as in article 163, where the flanges of each brace meet at one end, and diagonal lattice work forms the web. From (653),

Cross-section of a brace due wind, in square inches,

$$=S=\frac{2500l}{nB_1}=\frac{15l}{34n}.$$
 (654)

Increase this section by 50 per cent for the first and second braces, to take a part of load, as explained in article 187. See value of  $\Sigma y$ .

Weight of vertical braces, pounds,

$$= 2 \times \frac{5}{18} \times \frac{15l}{34n} \times 12\Sigma y$$

$$= 1.9607844 \left(1 + \frac{17}{n^2} - \frac{30}{n^3}\right) hl \qquad (655)$$

= hl	2.366729 2.291617 2.235294 2.192072 2.158224 2.131248	n = 8 $9$ $10$ $11$ $12$ $13$	=hl	2.091502 2.076631 2.064152 2.053578 2.044544 2.036764	n =	15 16 17 18 19 20
1	2.109415	14				

since we now have the special value

$$\sum y = \frac{4h}{n^2} \left\{ \frac{(n-1) + 2(n-2) + 3(n-3) + \dots (n-1) \text{ terms}}{+ (n-1) + 2(n-2) \text{ for suspenders}} \right\}$$

$$= \frac{2}{3}h \left( n + \frac{17}{n} - \frac{30}{n^2} \right).$$

194. The Head Lateral System. — Cross-section of head diagonals is given by equations (587), (588). Weight of head diagonals found in (589) and (590).

Cross-section of head strut expressed in (591) and (592).

Weight of head struts is given by (593) and (594).

Cross-section of iron to be added to segments of top chord shown in (595), (596).

Weight of added iron in (n - 6) panels of top chords is to be found in equation (597).

195. We may now collect the weights of all the parts of the bridge, and, after augmenting by one-tenth of itself the weight of the girders, the vertical braces, and the head struts, as explained in article 165, we may equate the weight so found to 2000nW, and so determine W in terms of L, l, and h, for different values of n, and from  $\frac{dW}{dh} = 0$  we find h rendering W a minimum.

*z* = 8.

 $+ h^2(16.31739L + 0.0116334l^2 + 2.603402l + 111.36l + 344.09l^{-1})$ ; (656)  $h[L(6.21262l + 567) + 0.067303l^{2} + 179.4758l] + 1.965105Ll^{2}$ W =

 $-1.70729247^2 + 16000h - 5.113284h^3$ 

l = 999.91 feet if  $h = l \div 10$ . l = 1673.77 feet if  $h = l \div 5$ , 2345.84 feet if h = l, Limiting span, ! =

n = 9

 $+ 0.059857l^2 + 179.5733l] + 2.188066Ll^2$ h[L(6.21604 + 648)]

W =

(657)  $+ h^{2}(20.13646L + 0.0112890)^{8} + 2.520779 + 130.987 + 451.01 - 1)$ 

l = 999.81 feet if  $h = l \div 10$ .  $-1.92070457^2 + 18000h - 5.771568h^3$ 

l = 1673.2 feet if  $h = l \div 5$ , 2340.01 feet if h = l, Limiting span, ! =

n = 10.

+ 0.053895<sup>2</sup> + 179.6513<sup>3</sup> + 2.4091155<sup>7</sup><sup>8</sup> h[L(6.2187l + 729)]

 $+ h^2(24.31409L + 0.0110482l^2 + 2.458823l + 151.125 + 565.71l^{-1})$ ; (658)

W =

l = 999.76 feet if  $h = l \div 10$ .  $-2.1341155l^2 + 20000h - 6.428092h^2$ = l, l = 1672.78 feet if  $h = l \div 5$ , 2335.85 feet if h = l, Limiting span, ! =

n = 11.

 $+ h^{2}(28.8519L + 0.0107913)^{2} + 2.411279J + 170.7436 + 699.85J^{-1};$  (659)  $-2.34752767^2 + 22000h - 7.083296h^2$  $+ 0.0490127l^2 + 179.7150l + 2.628778Ll^8$ h[L(6.2209/ + 810)]W =

l = 999.71 feet if  $h = l \div 10$ . l = 1672.46 feet if  $h = l \div 5$ , 2332.78 feet if h = l, Limiting span, ! =

n = 12

 $+ 0.044942l^2 + 179.7683l + 2.847397Ll^2$ h[L(6.222777 + 891

2.3740467 + 190.895 + 841.75/-1); (660)  $\frac{-2.5609386l^{2} + 24000l - 7.737532h^{2}}{2330.44 \text{ feet if } h = l, \quad l = 1672.22 \text{ feet if } h = l \div 5,$ + 43.751137 + 0.010602672 + W =

Limiting span, ! =

l = 999.67 feet if h = l + 10.

 $\frac{1}{2}$ : (663)

n = 13.

 $+ h^{2}(39.01278L + 0.0104075/^{2} + 2.344373/ + 210.5148 + 1003.11/^{-1});$  (661)  $-2.7743507l^{2} + 26000h - 8.39102h^{2}$  $+ 0.0414957^2 + 179.81337] + 3.06521621^3$ h[L(6.2243/+972)]W =

l = 99965 feet if  $h = l \div 10$ . l = 1672.03 feet if  $h = l \div 5$ , '/ = 2328.63 feet if h Limiting span, ! =

n = 14

 $+ h^{2}(44.6375L + 0.0102569l^{2} + 2.320356l + 230.662 + 1172.18l^{-1}); (662)$   $-2.9877628l^{2} + 28000h - 9.043848h^{2}$  $h[L(6.22567 + 1053) + 0.03853997^2 + 179.85197] + 3.282405L7^2$ 

W =

l = 999.62 feet if  $h = l \div 10$ . 2327.2 feet if h = l, l = 1671.89 feet if  $h = l \div 5$ , Limiting span, ! =

n = 15.

 $+ h^{2}(50.62606L + 0.0101045J^{2} + 2.30065J + 250.288 + 1360.80J^{-1})$ 1) + 0.035971<sup>2</sup> + 179.8853<sup>3</sup> + 3.49999<sup>2</sup><sup>18</sup> h[L(6.22687 + 1134

*W* ==

 $-3.2011738^{2} + 30000h - 9.69628h^{2}$ = l, l = 1671.77 feet if  $h = l \div 5$ ,

l = 999.61 feet if  $h = l \div 10$ . 2326.04 feet if h = l, 11 Limiting span, 1

n = 16

 $h[L(6.22784 + 1215) + 0.0337347^2 + 179.91454] + 3.71536527^2$ 

=M

 $+ h^{2}(56.9788L + 0.0099827l^{2} + 2.28429l + 270.422 + 1557.15l^{-1});$  (664)  $-3.41458487^2 + 32000h - 10.348272h^2$ 

l = 999.59 feet if  $h = l \div 10$ . l = 1671.67 feet if  $h = l \div 5$ 2325.1 feet if h = l, Limiting span, l =

n = 17.

 $h[L(6.2287l + 1296) + 0.031754l^2 + 179.9404l] + 3.93105Ll^2$ 

; (665)  $+ h^2(63.6961L + 0.009861l^2 + 2.27057l + 290.062 + 1772.93l^{-1})$  $-3.6279969/^{2} + 34000h - 11h^{2}$ W =

l = 999.58 feet if  $h = l \div 10$ . l = 1671.58 feet if  $h = l \div 5$ , 2324.31 feet if h = l, Limiting span, ! =

 $+ h^{2}(70.7783L + 0.0007613l^{2} + 2.258936l + 310.181 + 1996.49l^{-1});$  (666)  $-3.841409l^{2} + 36000l - 11.65138h^{2}$ ·|· l = 909.57 feet if h = ll = 1671.52 feet if  $h = l \div 5$ ,  $N[L(6.2295J + 1377) + 0.029994I^2 + 179.9633J] + 4.146965LI^2$ n = 18. 2323.66 feet if h = l, Limiting span, ! = W =

 $+ h^{2}(78.2259L + 0.0096616l^{2} + 2.248998l + 329.833 + 2239.49l^{-1})$  $-4.05482l^2 + 38000h - 12.30284l^2$  $h[L(6.2302l + 1458) + 0.028413l^2 + 179.9838l] + 4.362385Ll^2$ n = 19M = M

l = 999.56 feet if  $h = l \div 10$ . l = 1671.46 feet if  $h = l \div 5$ , 2323.08 feet if h = l, Limiting span, ! =

n = 20.

; (668)  $+ h^{2}(86.03828L + 0.0095784l^{2} + 2.24044l + 349.938 + 2490.27l^{-1})$  $-4.268232l^2 + 40000h - 12.9536h^3$  $h[L(6.2308l + 1539) + 0.027l^3 + 180.0023l] + 4.577607Ll^3$ W =

= 999.55 feet if  $h = l \div 10$ . l = 1671.41 feet if  $h = l \div 5$ 2322.64 feet if h = l, Limiting span, ! =

If a relation between W and L be assumed, we may substitute for L its value in terms of W, and find W and the The limiting spans are found by placing each denominator equal to zero, and replacing h by its assigned value. limiting span as before.

Tons.   Best number of panels   1.00   Best central height   1.00   Least bridge weight   1.00   Least bridge weight   1.00   Top chords,   Equation (636)   1.00   Do. head system,   Equation (636)   1.00   Bottom chords,   (649)   1.00   Girder diagonals,   (649)   1.00   Floor,   1.00   Floor,   (649)   1.00   Floor,   (649)   1.00   Floor,   (649)   1.00   Floor,   (649)   (575)   1.00   Floor,   (576)   (576)   1.00   Floor,   (576)   1.00   Floor,   (576)   1.00   Floor,   (576)   1.00   Floor,   (576)   1.00   Floor,   (576)   (576)   1.00   Floor,   (576)   1.00   Floor,   (576)   1.00   Floor,   (576)   1.00   Floor,   (576)   1.00   Floor,   (576)   (576)   1.00   Floor,   (576)   (576)   1.00   Floor,   (576)   (576)   (576)   (576)   (576)   (	Equation (636) (537) (637) (648) (649)	*			3			<b>4</b> 00			99	
	ation (636) (597) (637) (652) (648) (648)	*	<u>-</u>									
				22	23	<b>1</b>	7	13	16	10	18	17
			feet	40.325	39.726	39.134	83.584	82.871	82.154	134 292	133.514	132.781
	(636) (597) (637) (652) (648) (649)	Z Z	tons	106.234	106.157	106.285	455.119	454.685	455.032	1,245.722	1,244.937	1,245.175
	(597) (637) (652) (648) (648)		lbs.	44,943	45,380	45,857	245,456	246,683	248,165	762,604	764,945	767.584
	(637) (652) (648) (649)		lbs.	217	503	164	4,195	4,059	3,944	14,798	14,429	13,934
	(65a) (648) (649)		lbs.	27,844	28,257	28,697	150,049	151,264	152,647	453,556	456,003	458,580
	(648)		lbs.	13,917	13,595	13,311	113,720	112,041	110,537	408,511	404,194	399,991
	(646)		lbs.	22,219	23,220	24,20I	101,117	105,789	110,417	251,807	263,882	275,978
	(444)		lbs.	34,667	34,667	34,667	69,333	69,333	69,333	104,000	104,000	104,000
	15/5/		lbs.	21,733	20,062	18,628	74,514	69,547	65,200	456,480	146,700	138,070
	(226)		lbs.	16,944	17,000	17,047	35,496	35,488	35,483	55,204	55,081	54,968
	512), (577)		lbs.	3,201	3,484	3,741	7,745	8,338	8,888	13,361	14,272	15,232
	(509), (578)		lbs.	5,222	5,182	5,194	23,955	23,063	22,406	69,466	66,387	63,571
	(655)		lbs.	19,147	18,627	18,161	77,578	76,263	75,066	185,375	182,991	180,897
	589), (590)		js.	562	298	638	2,942	2,965	3,008	8,930	8,912	8,812
0 0 0 0 0 0 0 11 11 11 11 11 11 11 11 11	593), (594)		lbs.	1,552	1,738	1,937	4,138	4,537	4,970	7,353	8,078	8,733
	f-foot		lbs.	1,062	1,00,1	1,063	2,275	2,273	2,275	4,152	4,149	4,150
0.0 0.0 0.1 1.1 1.0 0.2	•	1 × + 1	<u>z</u>	164	15151	143	28\$	263	8	\$	372	35.17
0 0 0 0 11 11 11 11 19	•	1 + 4		4.960	5.034	5.111	4.786	4.827	4.869	4.468	<del>+</del> 494	4.519
0 0 0 10 10 10 10 10 10 10 10 10 10 10 1	•	W÷L		0.5311	0.5307	0.53r4	1.1378	1.1367	1.1376	2.0762	2.0749	2.0753
1.0 Bridge weight 1.5 Best number of panels { I.5 Best number of panels } 1.5 Best central height 1.5 Least bridge weight 2.0 Best number of panels }	<u> </u>	7+11		0.3469	0.3467	0.3470	0.4765	0.4763	0.4765	0.6749	0.6748	0.6748
1.5 Best number of panels 1.5 Best central height 1.5 Least bridge weight 2.0 Best number of panels 1		*		18	19	08						
1.5 Best number of panels 1.5 Best central height 1.5 Least bridge weight 2.0 Best number of panels 1.5		N K	tons	126.598	126.529	126.828					•	
1.5 Best central height 1.5 Least bridge weight 2.0 Best number of panels	•	×			13	14	13	14	18	14	18	16
2.0 Best number of panels 1	simultane-	*	<u>¥</u>	44.461	43.694	42.940	92.190	91.134	90.140	145.819	144.694	143.607
2.0 Best number of panels,	'sno	n W t	tons	135.660	135.647	135.922	579.238	578.833	579.187	1,555.091	1,554.032	1,557.373
		*		12	13	14	33	13	14	13	14	18
~_	simultanc-		leet Get	47.273	46.375	45 496	98.993	199.76	96.323	155.062	153.512	152.085
30 2.0 Least bridge weight ) U	(spo	7 7 2	tons	164.603	164.555	164.998	700.735	699.353	699.692	1,855.156	1,853.540	1,854.922

196. A few simple relations may be stated here, as resulting from our investigation of this bridge having two parabolic bowstring girders with double triangular web.

1st, For a given uniform live load,

$$n \propto l^{\frac{1}{4}}$$
 nearly, (669)

$$W \propto lh$$
 nearly. (670)

2d, For different live loads,

$$hW \propto nL$$
 nearly. (671)

197. EXAMPLE. — Specifications for the bridge of 200 feet span, as tabulated in article 194. We have found

$$l = 200$$
 feet,  $n = 13$ ,  $L = 15.38460$  tons;  $h = 39.726$  feet,  $W = 8.16590$  tons;  $q = 16$  feet,  $\frac{1}{2}(W + L) = 11.77525$  tons;  $q = 18$  feet.

Maximum horizontal component of chord strain, each of 2 girders,

$$= H = \frac{M}{y} = \frac{11.77525 \times 13 \times 200}{8 \times 39.726} = 96.333 \text{ tons,}$$

by equation (564). Therefore, by (632),

Strains in top chord due loads

$$=P=\frac{H}{\cos \alpha}=$$
 119.464 tons,  $\therefore$  Sections = 31.739 square inches;  
= 110.330 tons, = 29.312 square inches;  
= 107.231 tons, = 28.489 square inches;  
= 102.605 tons, = 27.260 square inches;  
= 99.170 tons, = 26.347 square inches;  
= 97.050 tons, = 25.783 square inches;  
= U = 96.333 tons, = 25.593 square inches;

for each half-span of each girder.

Additional cross-section of n-6 central panels of top chords to resist lateral displacement, by (595), is

```
S = 0.70049 square inch.
```

Corresponding chord strain =  $0.70049 \times 3.764 = 2.637$  tons.

```
Total top chord strains = 119.464 tons, 1st and 13th panels;
= 110.330 tons, 2d and 12th panels;
= 107.231 tons, 3d and 11th panels;
= 105.242 tons, 4th and 10th panels;
= 101.807 tons, 5th and 9th panels;
= 99.687 tons, 6th and 8th panels;
= 98.970 tons, 7th panel.
```

From (651), the varying sections of a bottom chord due wind are

```
S = \frac{2500 \times 39.7265 \times 200}{20000 \times 13^2 \times 18} (12, 2 × 11, 3 × 10, etc.)

= 3.918 square inches, ... Strains = 19.590 tons;

= 7.183 square inches, = 35.915 tons;

= 9.794 square inches, = 48.970 tons;

= 11.753 square inches, = 58.765 tons;

= 13.059 square inches, = 65.295 tons;

= 13.713 square inches, Add 96.333 tons,
```

for total sections and for total strains.

Putting  $\frac{1}{2}L$  for L in (647), we have

The cross-section of any girder diagonal

```
= \frac{3 \times 40000 \sec \theta}{64 \times 13 \times 39.7265} = 1.8153 \sec \theta.
1ST \text{ System.} \qquad 2ND \text{ System.}
2d panel = 2.250 square inches = 3.041 square inches,
3d panel = 3.791 square inches = 3.041 square inches,
4th panel = 3.791 square inches = 4.387 square inches,
5th panel = 4.795 square inches = 4.387 square inches,
6th panel = 4.795 square inches = 5.001 square inches,
7th panel = 5.001 square inches = 5.001 square inches.
```

The actual strains on these girder diagonals given in the diagram below, Fig. 120, where compressive strains are marked negative, have been derived from equations (641) and (642), using the proper value of r, and dividing by the proper value of  $\cos \theta$ .

The floor and the longitudinal I-beams will be the same here as in article 184; viz.,—

Floor of 21-inch oak.

I-beams, depth d = 9.7740 inches.

$$d - d_i = 0.6610$$
 inch.  
 $d_i = 9.1130$  inches.  
 $S = 5.0154$  square inches.  
 $I = 75.4160$ .

Also, the cross-section of the transverse I-beams due load will be, as in article 184,

S = 23.6108 square inches.

But, for the wind pressure,

$$W_1 = \frac{2500}{13} \times 39.7265 = 7639.7 \text{ pounds};$$

(511),  $S = \frac{2 \times 7639.7}{3 \times 13 \times 7542.5} (12^2, 11^2, 10^2, 9^2, 8^2, 7^2)$ 

= 7.480 square inches,  $\therefore$  Total = 31.091 square inches;
= 6.285 square inches, = 29.896 square inches;
= 5.194 square inches, = 28.805 square inches;
= 4.207 square inches, = 27.818 square inches;
= 3.324 square inches, = 26.935 square inches;
= 25.545 square inches, = 26.156 square inches;

for each half-span.

Take depth of beam d = 12 inches.

$$d - d_1 = 2$$
 inches.  
 $d_1 = 10$  inches.

Use 2 I-beams at each joint.

```
Then, by (558) and (557),
         Flange, b = 0.17803S = 5.5351 inches;
                                     = 5.3224 inches;
                                     = 5.1282 inches;
                                     = 4.9525 inches;
                                     = 4.7953 inches;
                                     = 4.6566 inches;
                  b_1 = 0.16363S = 5.0874 inches;
                                     = 4.8919 inches;
                                     = 4.7134 \text{ inches};
                                     = 4.5519 inches;
                                     = 4.4074 \text{ inches};
                                     = 4.2799 inches;
         Web, \delta - \delta_1 = 0.01440S = 0.4477 inch;
                                     = 0.4305 \text{ inch};
                                     = 0.4148 \text{ inch};
                                     = 0.4006 inch;
                                     = 0.3879 \text{ inch};
                                     = 0.3767 inch.
```

From (508), the cross-section of each horizontal diagonal in floor system is found, thus:

```
\sin \phi_1 = 0.76017,
S = \frac{7639.7}{15000 \times 2 \times 13 \times \sin \phi_1} (13 \times 12, 12 \times 11, 11 \times 10, \text{ etc.}),
= 4.020 \text{ square inches, 1st and 13th panels;}
= 3.402 \text{ square inches, 2d and 12th panels;}
= 2.835 \text{ square inches, 3d and 11th panels;}
= 2.319 \text{ square inches, 4th and 10th panels;}
= 1.855 \text{ square inches, 5th and 9th panels;}
= 1.443 \text{ square inches, 6th and 8th panels;}
= 1.082 \text{ square inches, 7th panel.}
\text{Cross-section of head diagonals is, from (587),}
S = \frac{0.0625 \times 168 \times 39.7265}{6 \times 13^3 \times 18 \times \cos \phi_1} = 0.54115 \text{ square inch,}
\text{since } \cos \phi_1 = 0.64972.
```

Cross-section of each head strut is given by (591),

$$S = \frac{168 \times 39.7265}{12 \times 169} = 3.291$$
 square inches.

Section of a brace, by (654), =  $S = \frac{15 \times 200}{34 \times 13} = 6.7873$  square inches.

Add 50 per cent,

For 2 end braces,

3.3936 square inches.

10.1809 square inches.

#### RESULTS.

Use, in each top chord, 2 10-inch channels, 21 square inches; I 15-inch plate by 0.76 to 0.36 inch; I 15  $\times$  18  $\times$  1 inch plate in every 3 feet, riveted to the bottom flanges. Make bottom chords of flat bars.

```
1st panel, 4 bars, 6 × 0.966 inch;

2d panel, 5 bars, 6 × 0.882 inch;

3d panel, 6 bars, 6 × 0.807 inch;

4th panel, 5 bars, 6 × 1.034 inches;

5th panel, 6 bars, 6 × 0.898 inch;

6th panel, 5 bars, 6 × 1.100 inches;

7th panel, 6 bars, 6 × 0.916 inch;
```

and proportion eyes as already specified.

In girder diagonals, use 4 angle irons  $2\frac{1}{4} \times 2\frac{1}{4}$  inches, latticed both ways by diagonal strips of wrought-iron  $1\frac{1}{4} \times \frac{1}{4}$  inch, and placed so far apart that the ratio of length to radius of gyration shall be 100, as already provided. This will require, for a strut of 40 feet length, a diameter of about  $\frac{40 \times 12}{100} \times 2$  = 9.6 inches, since by this arrangement of the material the radius of gyration is nearly one-half of the diameter.

In this case, where two struts intersect in a panel, the smaller one may pass within the flanges of the larger one at the intersection, involving thereby a little riveting in place, and perhaps a little irregularity of the lattice work. The pin

bearings at the ends of these diagonals are to be formed of wrought-iron plates affording a bearing-surface equal to

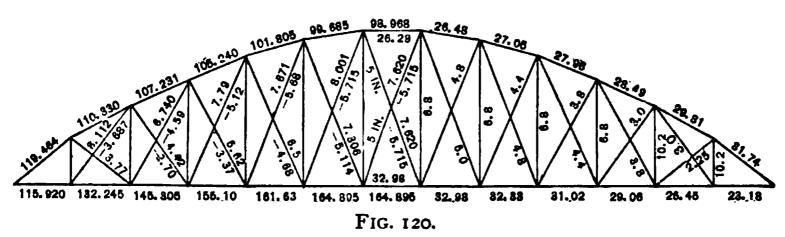
$$S = \frac{P_{\rm r}}{12000} = 2t \times d, \qquad (672)$$

where  $t = \frac{S}{2d}$  = thickness of plate, in inches;  $P_1$  = whole pressure on strut, in pounds; d = diameter of pin, in inches; as by general specifications.

It is here assumed, as in previous examples, that all crosssections can be made exactly as the calculations require; hence we need only notice further the head struts and side braces.

For head struts, use 2 light 4-inch channels latticed, giving the required section 3.291. For wind braces, use 2 7-inch channels latticed with a slope of 1 to 10; and, if a clear roadway of more than 10 (=18-4-4) feet is required, these braces must have their broad end at the top, or else they must have a bearing at the bottom beyond the ends of the transverse I-beams. Hence this mode of bracing high girders on narrow bridges is objectionable, and we shall henceforth either use a different style of brace, or provide it a head bearing, or increase the space between girders.

Parabolic Bow. — Two Girders.



Span, 200 feet. Central height, 39.726 feet (best). Maxima strains in each girder. Cross-sections in square inches. Live load, 1 ton to the running-foot. Bridge weight, 106.157 tons (minimum). Section of bottom chord pins for shearing = 3.3 to 3 inches. Section of bottom chord pins for bending = 5.2 to 4 inches. Diameter of bottom chord pins = 3\frac{3}{2}\$ to 3 inches. Diameter of top chord pins = 3\frac{3}{2}\$ inches. By equations (169), (46), and (52).

The deflection is derived from (559), putting  $B_1 = \frac{3.7646 + 5}{2}$  = 4.3823 tons, E = 12,000 tons,  $h_1 = h = 39.7265$  feet,  $a = \frac{1}{2}l$  = 100 feet, and measuring x from centre of span.

Deflection  $D_1 = 1.529$  inches for x = 0 at centre;  $D_2 = 1.522$  inches for  $x = 100 \div 13$ ;  $D_3 = 1.469$  inches for  $x = 300 \div 13$ ;  $D_4 = 1.361$  inches for  $x = 500 \div 13$ ;  $D_5 = 1.192$  inches for  $x = 700 \div 13$ ;  $D_6 = 0.836$  inch for  $x = 900 \div 13$ ;  $D_7 = 0.598$  inch for  $x = 1100 \div 13$ ;  $D_8 = 0$  inch for x = 100 at end.

Length of top chord =  $\frac{200}{13} \times \sum \sec \alpha = 219.297$  feet.

Contraction due strain, by (336),

$$=\lambda = \frac{3.7646 + 5}{12000} \times 219.297 \times 12 = 1.922$$
 inches.

Mean excess per panel = 
$$\frac{1.922}{13}$$
 = 0.1478 =  $\frac{1}{1}$  inch nearly.

## SECTION 2.

The Post Truss with Parabolic Top Chord (Fig. 36).

198. Let our previous notation be continued as far as applicable; viz.,—

l = span, in feet.

h =height of girder at centre, in feet.

n = number of panels, counting on the bottom chord, and odd.

L = panel weight of uniform live load, in tons, given.

W =panel weight of bridge, in tons, to be determined.

Symmetry here requires an odd number of panels for the bottom chord, and an even number for the top chord. As the

live load will here be applied to the bottom chord, we shall take n odd, and ranging from 9 to 21, inclusive. Each upper apex is in the middle of a panel's length. There is but a single system of counter diagonals, while there are two systems of mains.

We shall here assume the difference of level between the centre and end of the top chord to be one-tenth of the whole central height, h; and consequently the height at the end of a top chord is  $\frac{9}{10}h$ . This, of course, is wholly arbitrary, except, possibly, in some cases where the head room at the ends would be too little. The top chord is to be polygonal (that is, straight from joint to joint), and we will take it parabolic in this case.

Putting  $2 \times \frac{1}{10}h$  for h, and  $\frac{n-1}{n}l$  for l, and  $\frac{rl}{n}$  for x, in (472), we have the height of any upper apex,

$$y = h \left\{ 0.9 + \frac{0.4r(n-r-1)}{(n-1)^2} \right\} = \varepsilon h \text{ (say)}; (673)$$

r to be counted on top chord from o to  $\frac{n-1}{2}$ , inclusive (that is, to the centre).

# =	9	11	18	15	17	19	21
r = 0	0.90000	0.900	0.900000	0.900000	0.900000	0.900000	0.900
1	0.94375	0.936	0.930556	0.926531	0.923439	0.920988	0.919
2	0.97500	0.964	0.95556	0.948980	0.943750	0.939506	<b>o</b> .936
3	0.99375	0.984	0.97 5000	0.967347	0.960938	0.955556	0.951
4	1.00000	0.996	0.988889	0.981633	0.975000	0.969136	0.964
5		1.000	0.997222	0.991837	0.985938	0.980247	0.975
6			1.000000	0.997959	0.993750	0.988889	0.984
7				1.000000	0.998437	0.995062	0.991
8					1.000000	0.998765	0.996
9						1.000000	0.999
ŢΩ							7 000

Values of  $\varepsilon$  in (673).

Since the top chord for every panel length slopes uniformly, we have manifestly the height of girder, if measured in the vertical through any lower apex, equal to the mean of the two heights at the adjacent upper apices just found.

If, then, we put r + 1 for r in (673), and add the resulting equation to (673), we have, after dividing by 2,

$$y = h \left\{ 0.9 + \frac{0.4r(n-r-2) + 0.2(n-2)}{(n-1)^2} \right\} = \epsilon_i h, \quad (674)$$

which is the height through any lower apex; r taking the values 0, 1, 2, 3, ...  $\frac{n-3}{2}$ , counted on upper apices.

17 15 11 18 19 21 **#** = 0.91 5278 0.921875 0.913265 0.911718 0.918 0.910494 0.9095 0.943056 0.959375 0.950 0.937755 0.933595 0.930247 0.9275 0.965278 0.984375 0.974 0.958163 0.947531 0.9435 0.952344 0.996875 0.981944 0.967969 0.962346 0.990 0.974490 0.9575 0.980469 0.986735 0.974691 **0.**998 0.9695 0.993055 0.989844 5 6 0.998611 0.994898 0.984568 0.9795 0.998980 0.996094 0.991975 0.9875 7 0.999218 0.996913 0.9935 0.999382 8 0.9975 9 0.9995

VALUES OF & IN (674).

Calling  $\alpha$  the slope of any segment of the top chord, we have

$$\tan \alpha = \Delta y + \frac{l}{n} = \frac{0.4 \ln n}{(n-1)^2 l} [n-2(r+1)],$$

since 
$$\Delta y = y_{r+1} - y_r = \frac{0.4h}{(n-1)^2}[n-2(r+1)].$$

$$\sec^{2} \alpha = 1 + \frac{0.16h^{2}n^{2}}{(n-1)^{4}l^{2}}[n-2(r+1)]^{2}, \quad (675)$$

$$= 1 + \frac{h^{2}}{l^{2}}e_{2},$$

where r is to be counted 0, 1, 2, 3, etc., and

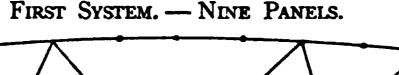
$$s_2 = \frac{0.16n^2}{(n-1)^4} [n-2(r+1)]^2.$$

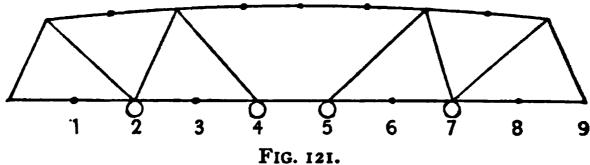
Values of  $e_2$  in (675).

n =	9	11	18	15	17	19	21
r = 0 1 2 3 4 5 6 7 8 9	0.155039 0.079102 0.028477 0.003164	0.156816 0.094864 0.048400 0.017424 0.001936	0.157785 0.105625 0.063896 0.032600 0.011736 0.001304	0.113390 0.075906	0.158752 0.119241 0.085374 0.057151 0.034573 0.017639 0.006350 0.000706	0.159014 0.123799 0.092987 0.066577 0.044568 0.026961 0.013755 0.004952 0.000550	0.159201 0.127449 0.099225 0.074529 0.053361 0.035721 0.021609 0.011025 0.003969 0.000441

199. Moments due a Total Dead Load of Uniform Panel Weight, W + L.—Although the total load is here uniform, the separate or single systems are in no case uniformly loaded throughout the girder's length; and we may find, by equations (40) and (43), the effect of each single weight, W + L, at all required points in each single system, or we may sum the values of a' in these two equations for the several cases, as follows:—

1st, When  $\frac{n-1}{4}$  is an integer; that is, when n=9, 13, 17, 21, etc.





Let r denote the number of intervals, each =  $2c = \frac{2l}{r}$ , between any weight in the right half-span and the left end of the girder.

Then, when  $x \bar{7} a'$ , equation (43) applies, giving

$$M=\frac{W+L}{l}a'(l-x).$$

Beginning at the left weight, and summing the values of d, we have

$$\Sigma a' = 2c \left\{ \left( 1 + 2 + 3 + \dots + \frac{n-1}{4} \right) + \left( \frac{n+1}{4} + \frac{n+5}{4} + \frac{n+9}{4} + \dots r \right) \right\}$$

$$= c \left\{ \frac{1}{16} (n+3) (n-1) + \left( r + \frac{n+1}{4} \right) \left( r - \frac{n-3}{4} \right) \right\};$$

$$\therefore M = \frac{W+L}{n} \left\{ \frac{n}{4} + r(r+1) \right\} (l-x), \tag{676}$$

which is the moment due all the weights on the length, 2cr, measured from the left end, at any point not distant less than 2cr from the left end of girder.

$$r \neq \frac{n-3}{4}$$

For the moments due the weights on the remaining part, l-2cr, we sum equation (40), where now  $M=\frac{W+L}{l}(l-a')x$ , and  $x \in a'$ .

Beginning at the point 2c(r + 1), we thus sum:

$$\sum a' = 2c[(r+1) + (r+2) + (r+3) + \dots r_1]$$

$$= c(r_1 + r + 1)(r_1 - r),$$

 $\Sigma a'^{\circ} = r_{i} - r = \text{number of terms};$ 

$$\therefore M = \frac{W + L}{n} [n(r_1 - r) - (r_1 + r + 1)(r_1 - r)]x$$

$$= \frac{W + L}{4n} (n - 2r - 2)(n - 2r)x, \qquad (677)$$

since here  $r_{\rm r} = \frac{n-2}{2}$ .

Adding (676) to (677) gives

$$M = \frac{W + L}{n} \left\{ \left[ \frac{n}{4} + r(r+1) \right] (l-x) + \frac{1}{4} (n-2r-2)(n-2r)x \right\}, \quad (678)$$

which is the moment due all weights in the first system, at any point in the second half-span, when  $\frac{n-1}{4}$  is an integer; and for the panel points in this system, second half-span, r becomes

$$\frac{n+1}{4}$$
,  $\frac{n+5}{4}$ ,  $\frac{n+9}{4}$ , etc.,

and

$$x=\frac{2rl}{n}=2cr.$$

Therefore, putting this value of x in (678),

$$M = \frac{(W+L)l}{4n}(2r+1)(n-2r), \qquad (679)$$

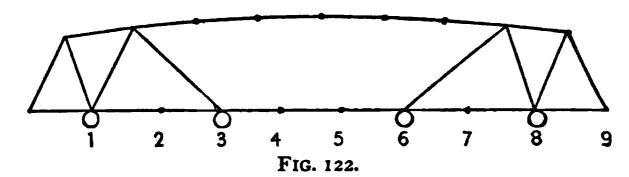
which is the moment due all the first system weights at loaded points in the second half-span.

If, in (678), 
$$x = 2(r + \frac{8}{4})\frac{l}{n}$$
, it becomes
$$M = \frac{(W+L)l}{8n} [4(n-2r-4)r + 5n - 9], (680)$$

which is the moment due all weights at the unloaded panel points in second half-span, first system.

Similarly we proceed with the second system when  $\frac{n-1}{4}$  is an integer.

SECOND SYSTEM. - NINE PANELS.



Beginning at the left weight, and summing the values of a' in (43), there results

$$\sum a' = 2c \left\{ \left( \frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \dots + \frac{n-3}{4} \right) + \left( \frac{n+3}{4} + \frac{n+7}{4} + \frac{n+11}{4} + \dots r \right) \right\}$$

$$= \frac{l}{n} \left\{ \left( \frac{n-1}{4} \right)^2 + \left( r + \frac{n+3}{4} \right) \left( r - \frac{n-1}{4} \right) \right\};$$

$$\therefore M = \frac{W + L}{l}(l - x)\Sigma a'$$

$$= \frac{W + L}{n}\left\{r(r + 1) - \frac{n - 1}{4}\right\}(l - x), \tag{681}$$

which is the moment due all the weights on the length, 2rc, measured from the left end of the girder, at any point not dis-

tant less than  $\frac{(n-3)l}{2n}$  from the left end; r being not less than  $\frac{(n-1)}{4}$ , and increasing by unity for the loaded points in second half-span.

For the remainder, l-2cr, we use (40), and find equation (677), which, since now  $r_1 = \frac{n-1}{2}$ , becomes

$$M = \frac{W + L}{4^n} (n - 2r - 1)^2 x, \qquad (682)$$

where x cannot be greater than 2c(r + 1), and r not less than  $\frac{n-1}{4}$ .

Adding (681) to (682), the result is, if, as usual, we call the sum M instead of 2M,

$$M = \frac{W + L}{n} \left\{ \left[ r(r+1) - \frac{n-1}{4} \right] (l-x) + \frac{1}{4} (n-2r-1)^2 x \right\}, \quad (683)$$

which is the moment due all weights in the second system, at any point in the second half-span, when  $\frac{n-1}{4}$  is an integer; the limits of x being 2rc and 2(r+1)c, and the limits of r,  $\frac{n-1}{4}$  and  $\frac{n-1}{2}$ .

If, in (683), we put  $x = \frac{2rl}{n}$ , we have, for the loaded points in second half-span, second system,

$$M = \frac{(W+L)l}{4^n} [(n-2r)(2r-1)+1]; \quad (684)$$

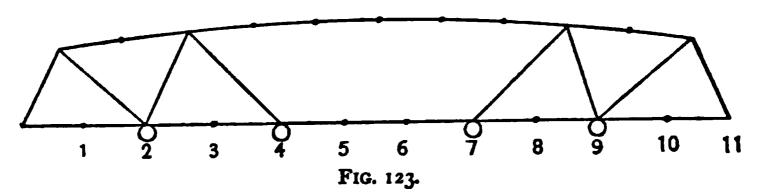
and, if  $x = 2(r + \frac{3}{4})\frac{l}{n}$ , (683) becomes

$$M = \frac{(W+L)l}{n} \left\{ r \left( \frac{n}{2} - r - 1 \right) + \frac{1}{8}(n-1) \right\}, \quad (685)$$

which is the moment at all upper or unloaded apices in second half-span, second system, the sign of the last moment to be changed from — to +.

2d, When  $\frac{n+1}{4}$  is an integer; that is, n=11, 15, 19, etc.

FIRST SYSTEM. - ELEVEN PANELS.



Proceeding, as above, to sum a' in equations (40) and (43), in the present case we write

$$M = \frac{W + L}{l}(l - x)\Sigma d'$$

$$= \frac{(W + L)}{n}(l - x)\left\{r(r + 1) - \frac{n}{4}\right\}, \quad (686)$$

since

$$\sum a' = 2c \left\{ \left( 1 + 2 + 3 + \dots + \frac{n-3}{4} \right) + \left( \frac{n+3}{4} + \frac{n+7}{4} + \frac{n+11}{4} + \dots r \right) \right\}$$

$$= \frac{l}{n} \left\{ r(r+1) - \frac{n}{4} \right\},$$

where  $x \bar{7} a'$ .

Again, when  $x \in a'$ ,

$$M = \frac{W + L}{l} \sum (la'' - a')x$$

$$= \frac{W + L}{n} (r_1 - r)(n - r_1 - r - 1)x, \quad (687)$$

since

$$\Sigma a^{\prime o} = r_i - r = \text{number of terms},$$

and

$$\Sigma a' = 2c[(r+1) + (r+2) + (r+3) + \dots r_1]$$

$$= \frac{l}{n}(r_1 - r)(r_1 + r + 1).$$

Add (687) to (686), and put  $r_i = \frac{n-2}{2}$ ;

$$M = \frac{W+L}{n} \left\{ \left[ r(r+1) - \frac{n}{4} \right] (l-x) + \frac{1}{4} (n-2r)(n-2r-2)x \right\}, \quad (688)$$

which is the moment due all weights in the first system, at any point in the second half-span, when  $\frac{n+1}{4}$  is an integer; r

being  $\frac{n-1}{4}$ ,  $\frac{n+3}{4}$ ,  $\frac{n+7}{4}$ , etc., and x lying between 2rc and 2(r+1)c.

If x = 2rc, equation (688) becomes

$$M = \frac{(W+L)l}{4n}(2r-1)(n-2r), \qquad (689)$$

which is the moment due all the first system weights at loaded points in the second half-span.

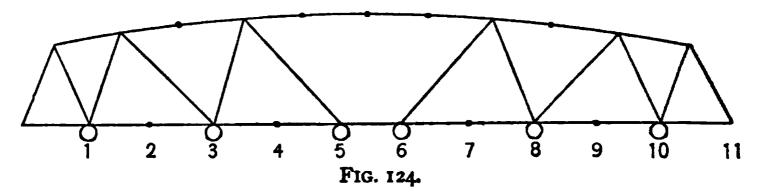
If, in (688),  $x = 2(r + \frac{3}{4})c$ , we have

$$M = \frac{(W+L)l}{n} \left\{ r \left( \frac{n}{2} - r - 1 \right) + \frac{1}{8}(n-3) \right\}, \quad (690)$$

which is the moment due all weights at the unloaded apices in the second half-span, first system.

Also, for the second system, when  $\frac{n+1}{4}$  is an integer, we write:—

SECOND SYSTEM. — ELEVEN PANELS.



From (43), we have

$$x = a'$$

$$\sum a' = 2c \left\{ \left( \frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \dots + \frac{n-1}{4} \right) + \left( \frac{n+1}{4} + \frac{n+5}{4} + \frac{n+9}{4} + \dots r \right) \right\}$$

$$= c \left\{ \left( \frac{n+1}{4} \right)^2 + \left( r + \frac{n+1}{4} \right) \left( r - \frac{n-3}{4} \right) \right\}$$

$$= c \left[ \frac{1}{4} (n+1) + r(r+1) \right],$$

$$\therefore M = \frac{W + L}{n} [\frac{1}{4}(n+1) + r(r+1)](l-x). \tag{691}$$

And, from (40),

$$x \in a', (r_1 - r) \text{ terms};$$

$$\Sigma a' = 2c[(r + 1) + (r + 2) + (r + 3) + \dots r_1],$$

$$-\Sigma a' + \Sigma a'^{\circ} = [-(r_1 + r + 1)(r_1 - r) + n(r_1 - r)]c$$
  
=  $\frac{1}{4}c(n - 2r - 1)^2$ 

if 
$$r_i = \frac{n-1}{2}$$
.

$$M = \frac{W + L}{4n}(n - 2r - 1)^2x.$$
 (692)

The sum of (691) and (692) is

$$M = \frac{W + L}{4^n} \{ [n + 1 + 4r(r + 1)](l - x) + (n - 2r - 1)^2 x \}, \quad (693)$$

which is the moment due all weights in the second system, at any point in the second half-span, when  $\frac{n+1}{4}$  is an integer; the limits of x being 2rc and 2(r+1)c, and the limits of r being  $\frac{n-3}{4}$  and  $\frac{n-1}{2}$ .

Substituting 2rc for x in (693), we get

$$M = \frac{(W+L)l}{4n}[n+1+r(2n-4r-2)], (694)$$

which is the moment due all weights at the loaded apices, second half-span, second system,  $\frac{n+1}{4}$  being an integer.

And, if  $x = 2(r + \frac{3}{4})c$  in (693), we have finally

$$M = \frac{(W+L)l}{4n} \left[ \frac{1}{2} (5n-7) + 2r(n-2r-4) \right], \quad (695)$$

which is the moment due all weights at all upper or unloaded apices in second half-span, second system; the sign of the last moment to be changed from — to +, as in equation (685).

Of course, the total moments for each and both systems will be equal at corresponding points in the two half-spans.

200. Weights of Top and Bottom Chords due a Total Dead Load of Uniform Panel Weight, W + L. — Dividing (679) by (674) gives

$$H = \frac{(W+L)l}{h} \times \frac{e_3}{\epsilon_1} \tag{696}$$

which is the horizontal component of strain in top chord over loaded points in first system if

$$\varepsilon_3 = \frac{(2r+1)(n-2r)}{4n},$$

and  $\frac{n-1}{4}$  is an integer.

Also, dividing (684) by (674), calling

$$\epsilon_4 = \frac{(n-2r)(2r-1)+1}{4n},$$

we get

$$H = \frac{(W+L)l}{h} \times \frac{\epsilon_4}{\epsilon_1}, \tag{697}$$

which is horizontal component of strain in top chord over loaded points in second system, for this case. Then, adding the strains due to each panel length of top chord from the two systems, we have total horizontal component of strain due n(W + L) in each panel of top chord,

$$H = \frac{(W+L)l}{h} \times \epsilon_5, \tag{698}$$

if  $\epsilon_5$  = the sum of the proper values of  $\frac{\epsilon_3}{\epsilon_1}$  and  $\frac{\epsilon_4}{\epsilon_1}$ 

Similarly, in case  $\frac{n+1}{4}$  is an integer, making

$$\varepsilon_3 = \frac{(n-2r)(2r-1)}{4n}$$

in equation (689), and

$$\varepsilon_4 = \frac{(n-2r)(2r+1)+1}{4n}$$

in (694).

Computing in this manner, we have, for each half-span,

		7	VALUES (	of e <sub>5</sub>	IN (	(698).
# =	9	11	18	18	5	17

# <b>=</b>	9	11	18	15	17	19	21
Panel Pt.							
I	0.704320	0.679863	0.741537	0.718544	0.762139	0.741653	0.775200
2	0.914764	1.036126	1.047237	1.122328	1.121732	1.174723	1.169531
3	1.120256	1.204315	1.341205	1.379895	1.468452	1.487771	1.551306
4	1.120256	1.370786	1.480524	1.629998	1.690715	1.791639	1.828413
5		1.370786	1.621512	1.749982	1.909621	1.988183	2,100001
6			1.621512	1.871858	2.014554	2.182537	2.275878
7			-	1.871858	2,122266	2.276187	2.451159
8					2.122266	2.372470	2.535517
9	in the second se					2.372470	2.622746
10	<u> </u>						2.622746
Σe <sub>5</sub>	7.719192	11.323752	15.707054	20,688926	26.423490	32.775266	39.864994

Strain on top chords 
$$= H \sec \alpha$$
  
Section of top chords  $= \frac{H \sec \alpha}{Q}$  (699)

Weight of top chords, in pounds, due (W + L)n

$$= \frac{5}{18} \times \frac{12l}{nQ} \Sigma H \sec^2 \alpha$$

$$= \frac{(W+L)l^2}{h} \times \frac{10}{3nQ} \Sigma \epsilon_5 \sec^2 \alpha$$

$$= \frac{W+L}{h} \times \frac{10}{3nQ} (l^2 \Sigma \epsilon_5 + h^2 \Sigma \epsilon_2 \epsilon_5), \qquad (700)$$

since, by (675),  $\sec^2 \alpha = 1 + \frac{h^2}{l^2} \epsilon_2$ .

Using the proper values of  $\varepsilon_2$  and  $\varepsilon_5$ , we find

VALUES	OF	<b>8</b> <sub>2</sub> <b>8</b> <sub>5</sub>	IN	(700)	).
--------	----	---	----	-------	----

# =	9	11	18	15	17 .	19	21
Panel Pt.							
1 2 3 4 5 6	0.109197 0.072376 0.031901 0.00\$545	0.106613 0.098291 0.058289 0.023632 0.002654	0.117003 0.110614 0.085697 0.048265 0.019030 0.002114	0.113797 0.127261 0.104742 0.074846 0.040999 0.015787	0.120991 0.133756 0.125368 0.096626 0.066021 0.035535	0.117933 0.145430 0.138343 0.119282 0.088609 0.058843	0.123413 0.149056 0.153928 0.136270 0.112058 0.081297
7 8 9 10				0.001754	0.013476 0.001498	0.031309 0.011748 0.001305	0.052967 0.027954 0.010410 0.001157
∑e₂e₅	0.434038	0.578958	0.765446	0.958372	1.186542	1.425604	1.697020

Hence, if we take, as in article 186, equation (634), Q = 3.7647 tons, we have finally

Weight of top chords due n(W + L), in pounds,

$$=\frac{W+L}{h}\times\frac{10}{3\times3.7647n}(l^2\Sigma\epsilon_5+h^2\Sigma\epsilon_2\epsilon_5)$$
,701)

$$= \frac{W + L}{h} \quad 0.759412l^{2} + 0.042701h^{2} \qquad n = 9$$

$$0.906448l^{2} + 0.046602h^{2} \qquad 11$$

$$1.069793l^{2} + 0.052134h^{2} \qquad 13$$

$$1.221223l^{2} + 0.056571h^{2} \qquad 15$$

$$1.376226l^{2} + 0.061799h^{2} \qquad 17$$

$$1.527358l^{2} + 0.066434h^{2} \qquad 19$$

$$1.680811l^{2} + 0.071551h^{2} \qquad 21$$

Again, dividing (680) by (673) gives

$$H = \frac{(W+L)l}{\hbar} \times \frac{\epsilon_6}{\epsilon} \tag{702}$$

for the bottom chord strain under an upper apex in the first system,  $\frac{n-1}{4}$  an integer, and  $\epsilon_6 = \frac{4r(n-2r-4)+5n-9}{8n}$ 

Also, dividing (685) by (673), and putting

$$\epsilon_7 = \frac{r}{n} \left( \frac{n}{2} - r - 1 \right) + \frac{n-1}{8n},$$

we have

$$H = \frac{(W+L)l}{h} \times \frac{\epsilon_7}{\epsilon} \tag{703}$$

as the bottom chord strain under an upper apex in the second system;  $\frac{n-1}{4}$  being an integer.

Then, adding the two strains thus found for each panel length of bottom chord, we find, for this case,

$$H = \frac{(W+L)l}{h} \times \epsilon_8, \qquad (704)$$

which is the total strain on bottom chord due n(W + L).

Proceed in like manner in case  $\frac{n+1}{4}$  is an integer, making

$$\epsilon_6 = \frac{r}{n} \left( \frac{n}{2} - r - 1 \right) + \frac{n-3}{8n}$$

in (690), and

$$\epsilon_{7} = \frac{r}{2n}(n-2r-4) + \frac{5n-7}{8n}$$

in (695).

/

Computing thus, we find for each half-span, including middle panel, —

<b>x</b> =	9	11	18	15	17	19	21
Panel.  2  3 4  5 6	0.246913 0.417791 0.807154 0.957264 1.111111	0.252525 0.489510 0.812868 1.117273 1.238360 1.363636	0.256410 0.458860 0.894161 1.155222 1.408483 1.509075 1.615385	0.259259 0.506825 0.887470 1.249844 1.471186 1.687720 1.774622	0.261438 0.481072 0.941122 1.264131 1.578340 1.770084 1.960186	0.263158 0.516987 0.932223 1.330849 1.613270 1.888460 2.058469	0.264550 0.494988 0.973214 1.336554 1.689301 1.940049 2.186634
7 8 9 10			1.015305	1.774022	2.036256 2.117647	2.227605 2.295612 2.368422	2.13003 2.33904 2.49180 2.55307 2.61904
$\Sigma \epsilon_8$	5.969355	9.184709	12.979807	17.540519	22.702905	28.621688	33.96698

Values of  $\varepsilon_8$  in (704).

All except the middle panel taken twice for \(\Sigma\_{8}\).

Taking T = 5 tons, the allowed inch strain in tension, as in (634), we find, from (704),

Cross-section of bottom chords, 
$$S = \frac{(W + L)l}{hT} \times \epsilon_8$$
. (705)

Weight of bottom chords due (W + L)n, in pounds,

201. To find the Greatest Strains in the Girder Diagonals, and their Weights. — Equation (148) gives the strain on the counter diagonals in this case in terms of the simultaneous moments in the vertical planes,  $AA_1$ ,  $BB_1$ , etc., Fig. 38. These moments let us compute for the uniform panel load, L tons, advancing over the panel points O, B, D, etc., of the horizontal bottom chord. The difference of the horizontal strains due to these moments at consecutive panel points will be greatest when the foremost panel weight of load is at the foot of a Y diagonal or counter. We use, therefore, the ordinary formulæ (64) and (68), giving simultaneous moments at O and O, etc.; then, for the moment at O, etc., we take one-half the sum of (64) and (68, thus:

$$M_{r+\frac{1}{2}} = \frac{Ll}{2n^2}r(r+1)(n-r-\frac{1}{2})$$

$$= Lle_9 \text{ (say)}, \tag{707}$$

which is the value of  $M_1$  in (148), and would also be obtained by putting  $r_2 = 0$ ,  $x = (r + \frac{1}{2})c = (r + \frac{1}{2})\frac{l}{n}$ , and L for W, in (60) for the half-intervals OA, BC, etc., Fig. 38.

 $M_{r+1}$  in equation (68) takes the place of  $M_2$  in (148).

$$h_1$$
 in (148) =  $AA_1$ , Fig. 38, =  $y$  in (673);  
and  
 $h_2$  in (148) =  $BB_1$ , Fig. 38, =  $y$  in (674).  
 $a_1 = \frac{1}{3}CC_1 = \frac{1}{3}y_{r+1}$ , (673).  
 $b_1 = 2a_1$ .

With these values of  $M_1$ ,  $M_2$ ,  $h_1$ ,  $h_2$ ,  $a_1$ ,  $b_1$ , we compute  $Y \cos \phi$  of (148), which, with sign changed, becomes

$$Y\cos\phi = \frac{Ll}{h}\epsilon_{10}.$$
 (708)

Values of $\varepsilon_{10}$ in (708)	VALUES	OF	8.0	IN	(708)	
---------------------------------------	--------	----	-----	----	-------	--

# <b>=</b>	9	11	18	15	17	19	21
Panel.  2  3 4 5 6 7 8 9 10	0.024108 0.063502 0.115315 0.180545	0.016761 0.044923 0.082009 0.127461 0.182225	0.012321 0.033488 0.061579 0.095784 0.136113 0.183280	0.009436 0.025928 0.048030 0.074941 0.106409 0.142579 0.183999	0.007455 0.020671 0.038533 0.060364 0.086241 0.114847 0.147601 0.184512	0.006038 0.016869 0.031615 0.049719 0.070841 0.094826 0.121698 0.121698 0.151622 0.184900	0.004992 0.014025 0.026415 0.041690 0.059543 0.079795 0.102391 0.127379 0.154894 0.185203
Σe <sub>10</sub>	0.766940	0.906758	1.045130	1.182644	1.320448	1-456256	1.592654

Strain on counter 
$$= Y = \frac{Ll}{h} \times \epsilon_{10} \sec \phi. \quad (709)$$

Cross-section of counter = 
$$S = \frac{Lle_{10}}{Th} \sec \phi$$
. (710)

Weight of (n - 1) counters, pounds,

$$= \frac{5}{18} \times \frac{12 \times 3^{l}}{2n} \times \frac{L^{l}}{5h} \Sigma \epsilon_{10} \sec^{2} \phi$$

$$= \frac{L^{l^{2}}}{nh} \Sigma \epsilon_{10} \sec^{2} \phi$$

$$= \frac{L}{h} \left( \frac{l^{2}}{n} \Sigma \epsilon_{10} + \frac{4nh^{2}}{9} \Sigma \epsilon_{10} \epsilon^{2} \right) \qquad (711)$$

$$= \frac{L}{h} \begin{vmatrix} 0.085216l^{2} + 3.033912h^{2} & n = 9 \\ 0.082433l^{2} + 4.386878h^{2} & 11 \\ 0.080395l^{2} + 5.973413h^{2} & 13 \\ 0.078843l^{2} + 7.794240h^{2} & 15 \\ 0.07673l^{2} + 9.855875h^{2} & 17 \\ 0.076645l^{2} + 12.140274h^{2} & 19 \\ 0.075841l^{2} + 14.665989h^{2} & 21 \end{vmatrix}$$

since we take T = 5 tons per square inch in tension, and

$$\sec^2 \phi = 1 + \frac{4n^2}{9l^2} y_{r+1}^2 = 1 + \frac{4n^2h^2}{9l^2} \epsilon^2.$$
 (712)

Manifestly  $\epsilon^2$  is to be taken from (673), always beginning with r=2.

Main Diagonals. — To find the strains, sections, and weights of the main diagonals of the Post truss with parabolic top chord, we proceed as follows:—

When  $\frac{n-1}{4}$  is an integer, use equation (676) for the live load, nL, making W=0; and for moment in second half-span, first-system apices,

At foremost end of live load, put

$$x=\frac{2rl}{n}$$
, giving  $M_o$ ;

At point 1½ panels ahead of foremost end, put

$$x=\frac{2(r+\frac{3}{4})l}{n}, \text{ giving } M_1;$$

At point 2 panels ahead of foremost end, put

$$x=\frac{2(r+1)l}{n}, \text{ giving } M_1;$$

these three moments being simultaneous. Then

$$M_{0} = \frac{Ll}{n^{2}} \left\{ \frac{n}{4} + r(r+1) \right\} (n-2r)$$

$$M_{\frac{3}{4}} = \frac{Ll}{n^{2}} \left\{ \frac{n}{4} + r(r+1) \right\} \left( n - 2r - \frac{3}{2} \right)$$

$$M_{1} = \frac{Ll}{n^{2}} \left\{ \frac{n}{4} + r(r+1) \right\} (n-2r-2)$$

$$(713)$$

In a similar manner, for the second half-span, second-system

apices, we find, from (681), simultaneous moments due live load, nL,

$$M_{0} = \frac{Ll}{n^{2}} \left\{ r(r+1) - \frac{n-1}{4} \right\} (n-2r)$$

$$M_{\frac{3}{4}} = \frac{Ll}{n^{2}} \left\{ r(r+1) - \frac{n-1}{4} \right\} (n-2r-\frac{3}{2})$$

$$M_{1} = \frac{Ll}{n^{2}} \left\{ r(r+1) - \frac{n-1}{4} \right\} (n-2r-2)$$

$$(714)$$

Dividing each of these moments, (713), (714), by the height, y, of truss at the section where the moment is taken, we find the horizontal strains at the panel points in second half-span,

At loaded points, 
$$H_0 = \frac{M_0}{y_0}$$
 from (713), (714), (674);  
At unloaded points,  $H_1 = \frac{M_1}{y_1}$  from (713), (714), (673);  
At unloaded points,  $H_1 = \frac{M_1}{y_1}$  from (713), (714), (674).

The difference of the two simultaneous horizontal strains at vertical sections through the ends of a diagonal at and next ahead of foremost end of live uniform load is the horizontal component of maximum strain on that diagonal due live load, and is tension on the diagonal whose foot is at the foremost end, but compression on the next.

$$\triangle H = H_0 - H_1 = \frac{Ll}{n^2h} \times \varepsilon_{11} \text{ (tension)},$$
 (715) 
$$\Delta H = H_1 - H_1 = \frac{Ll}{n^2h} \times \varepsilon_{12} \text{ (compression)};$$
 (716)

 $\epsilon_{11}$  and  $\epsilon_{12}$  being functions of n and r in (673), (674), (713), (714). For the moments due the dead load, nW, at the same points where the simultaneous moments due live load have been found,

 $\frac{n-1}{4}$  being an integer, we use equations (679) and (680), and (679) with r+1 for r, L=0, thus:—

First system,

$$M_{0} = \frac{Wl}{4n}(2r+1)(n-2r)$$

$$M_{1} = \frac{Wl}{4n}\left\{2(n-2r-4)r + \frac{5n-9}{2}\right\}. \quad (717)$$

$$M_{1} = \frac{Wl}{4n}(2r+3)(n-2r-2)$$

Second system, use (684) and (685),

$$M_{0} = \frac{W?}{4n} [(n-2r)(2r-1)+1)$$

$$M_{1} = \frac{W?}{4n} \left\{ 4r \left( \frac{n}{2} - r - 1 \right) + \frac{1}{2}(n-1) \right\}$$

$$M_{1} = \frac{W?}{4n} [(n-2r-2)(2r+1)+1]$$

Dividing each moment by the proper value of y from (673) and (674), we obtain horizontal strains at all required apices in each system, from the differences of which consecutive horizontal strains comes the horizontal component of maximum diagonal strain due dead load, thus:

$$\Delta H = H_0 - H_1 = \frac{W7}{4nh} \times \epsilon_{13} \text{ (tension)}, \qquad (719)$$

$$\Delta H = H_1 - H_{10} = \frac{Wl}{4nh} \times \epsilon_{14} \text{ (compression)}; (720)$$

 $e_{13}$  and  $e_{14}$  being functions of n and r in (673), (674), (717), (718).



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